Glasgow Math. J.: page 1 of 23.
doi:10.1017/S0017089516000094.

A TRACE FORMULA FOR HECKE OPERATORS ON VECTOR-VALUED MODULAR FORMS

NICOLE RAULF
Université de Lille 1, U.M.R. CNRS 8524, U.F.R. de Mathématiques,
Cité Scientifique, F-59655 Villeneuve d'Ascq Cédex
e-mail: nicole.raulf@math.univ-lille1.fr

and OLIVER STEIN
Ostbayerische Technische Hochschule Regensburg,
Fakultät für Informatik und Mathematik,
Universitätsstraße 31, 93053 Regensburg, Germany
e-mail: oliver.stein@hs-regensburg.de

(Received 25 August 2014; revised 27 April 2015; accepted 6 July 2015)

Abstract. We present a ready to compute trace formula for Hecke operators on vector-valued modular forms of integral weight for $\mathrm{SL}_2(\mathbb{Z})$ transforming under the Weil representation. As a corollary, we obtain a ready to compute dimension formula for the corresponding space of vector-valued cusp forms, which is more general than the dimension formulae previously published in the vector-valued setting.

2000 *Mathematics Subject Classification.* 11F27, 11F25, 11F72.

1. Introduction.

Trace formulae in general and especially trace formulae for Hecke operators on a space of modular forms are an important tool in the theory of modular forms. For instance, they provide the possibility to compute the dimension of the underlying space of cusps forms. The eigenvalues $\lambda(n)$ of the Hecke operators $T(n)$ and the Fourier coefficients $a(n)$ of cusp forms are fundamental objects and vital for the study of modular forms. An explicit trace formula allows to study the arithmetic properties of these eigenvalues and thereby the corresponding properties of the Fourier coefficients $a(n)$ of a simultaneous eigenform, given that there is a simple relation between eigenvalues and coefficients. Therefore, a trace formula also provides an approach to access the Fourier coefficients of cusp forms. Furthermore, the trace formula is useful to understand class numbers of binary quadratic forms. There are several versions of trace formulae for different types of modular forms and applications, see e.g. [**1**, **13**, **20**, **23**, **19**, **24**, **31**, **25**, **26**, **27**, **32**, **35**], and [**33**].

Vector-valued modular forms and more general vector-valued automorphic forms associated to the Weil representation play an important role in many recent papers see e.g. [**3**, **4**, **6**, **9**] and [**30**]. One reason for this is their connection to the so called singular theta lift, [**3**] and [**6**], which allows interesting applications in geometry and algebra.

The main goal of the present paper is to give a trace formula for Hecke operators on vector-valued modular forms for the Weil representation. It is intended to be explicit enough to be used on a computer. There are some papers which present a trace formula for vector-valued automorphic forms in a more general setting, see e.g. [**22**] or Hejhal's books [**17**] and [**18**]. However, the trace formulae given there cannot be simply quoted since they are either not directly applicable to our situation or not explicit enough.

Let us now describe the results of this paper: Let $(L, (\cdot, \cdot))$ be an even non-degenerate lattice of type (b^+, b^-) where $(\cdot, \cdot)$ denotes a bilinear form on L with associated quadratic form $x \mapsto x^2/2$, L' be the dual lattice of L and L'/L the discriminant group. For the introduction, we restrict ourselves to the case that the signature $b^+ - b^-$ of L is even. In this paper, we consider vector-valued modular forms which transform under a particular representation. This representation, defined by (2.2) and (2.3), is a representation of $\mathrm{SL}_2(\mathbb{Z})$ on the group ring $\mathbb{C}[L'/L]$,

$$\rho_L : \mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{GL}(\mathbb{C}[L'/L]),$$

which is an example of a so-called Weil representation. A modular form of weight $k \in \mathbb{Z}$ and type ρ_L for $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}[L'/L]$ which satisfies

$$f(\gamma\tau) = (c\tau + d)^k \rho_L(\gamma) f(\tau)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and which is holomorphic at the cusp ∞. The space of these modular forms is denoted by $M_{k,L}$ and the space of cusp forms by $S_{k,L}$.

There are several ways to establish a trace formula. Our approach is an adaption of the papers [**39**] and [**23**] to the situation of vector-valued modular forms of type ρ_L. One important step in these papers is the construction of a kernel function for the underlying space of cusp forms with respect to the Petersson scalar product. Here, we will show that

$$h_{\beta,1}(\tau, \tau') = \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} \phi(\gamma, \tau)^{-2k} (\gamma\tau + \tau')^{-k} \rho_L^{-1}(\gamma) \mathfrak{e}_\beta$$

is a kernel function for $S_{k,L}$ with respect to the Petersson scalar product, see Theorem 5.6. Then, similar to the papers above quoted, the trace of the Hecke operator $T(n)$ can be expressed in terms of the integral

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle h_{\beta,n}(\tau, -\overline{\tau}), \mathfrak{e}_\beta \rangle \, \mathrm{Im}(\tau)^k d\mu(\tau), \tag{1.1}$$

where $h_{\beta,n}$ is a slightly more general vector-valued function than $h_{\beta,1}$. These results are valid if the signature $\mathrm{sig}(L) = b^+ - b^-$ of L is even and also if $\mathrm{sig}(L)$ is odd.

The rest of the paper deals with the evaluation of (1.1), see the proof of Theorem 6.3. For these calculations we limit ourselves to the case of even signature of L because it seems that the case of odd signature involves tedious computations, which should be done in a separate paper. It turns out that the character of the Weil representation occurs in the trace formula. Therefore, an explicit expression for the character of the Weil representation is derived, see Theorem 3.1. The case $|L'/L| = p$, where p is a prime, is treated separately and the character of the Weil representation is described very explicitly in terms of the Kronecker symbol, see Theorem 3.3.

Finally, as a corollary, a dimension formula for the space $S_{k,L}$ is presented. As already mentioned more general trace formulae exist already in the literature which lead to more general dimension formulae than the one provided in the paper (see e.g. [**14**]). However, our dimension formula has the advantage that it applies directly to the special setting of vector-valued modular forms for the Weil representation. And as such it seems (to the best of our knowledge) to be a generalisation of the published dimension formulae for $S_{k,L}$, since it does not rely on the condition $2k \equiv -\operatorname{sig}(L) \pmod 4$. For other examples of dimension formulas in the vector-valued setting see [**4**] and [**7**].

It would be interesting to use the derived trace formula to calculate eigenvalues of Hecke eigenforms and to study their arithmetic properties. Moreover, it would be interesting to try to determine the distribution of the eigenvalues of Hecke operators which will be done in forthcoming papers.

2. The Weil representation and vector-valued modular forms. In this section, we introduce the notation that is used throughout this paper. Furthermore, we recall some basic facts concerning the Weil representation and vector-valued modular forms associated to the Weil representation where we essentially follow [**6**] and [**10**].

The group $\mathrm{GL}_2^+(\mathbb{R}) = \{M \in \mathrm{GL}_2(\mathbb{R});\quad \det(M) > 0\}$ acts on the upper half-plane $\mathbb{H} = \left\{\tau = x + iy \in \mathbb{C};\quad \operatorname{Im}(\tau) > 0\right\}$ and on $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$, respectively, by linear fractional transformations. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ and $\tau \in \mathbb{H}$ let $j(M,\tau) := \sqrt{c\tau + d}$ where we choose the principal branch of the square root. As is common, we introduce the double cover $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ of $\mathrm{GL}_2(\mathbb{R})$. This group is the set of pairs $(M, \phi(M,\tau))$ where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ and $\phi(M,\tau) = \det(M)^{-1/4} j(M,\tau)$ or $\phi(M,\tau) = -\det(M)^{-1/4} j(M,\tau)$. The multiplication of two elements $(M_1, \phi_1(M_1,\tau))$, $(M_2, \phi_2(M_2,\tau)) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ is given by

$$(M_1, \phi_1(M_1,\tau))(M_2, \phi_2(M_2,\tau)) = (M_1M_2, \phi_1(M_1, M_2\tau)\phi_2(M_2,\tau)).$$

A non-scalar element $M \in \mathrm{GL}_2^+(\mathbb{R})$ is defined to be elliptic, parabolic or hyperbolic if it satisfies

$$\operatorname{tr}(M)^2 < 4\det(M), \quad \operatorname{tr}(M)^2 > 4\det(M), \text{ or } \operatorname{tr}(M)^2 = 4\det(M).$$

Moreover, we call an element $\widetilde{M} = (M, \phi(M,\tau)) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ elliptic, hyperbolic or parabolic, respectively, if M has the corresponding property. If G is a subset of $\mathrm{GL}_2^+(\mathbb{R})$, we write $\widetilde{G}$ for its inverse image under the covering map $\widetilde{\mathrm{GL}}_2^+(\mathbb{R}) \to \mathrm{GL}_2^+(\mathbb{R})$. Throughout this paper $\Gamma(1)$ denotes the full modular group $\mathrm{SL}_2(\mathbb{Z})$. It is well known that the integral metaplectic group $\widetilde{\Gamma}(1)$ is generated by $T = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$, and $S = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right)$. One has the relations $S^2 = (ST)^3 = Z$ with $Z = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i\right)$ being the standard generator of the centre of $\widetilde{\Gamma}(1)$.

By $(L, (\cdot,\cdot))$ we denote an even non-degenerate lattice of type (b^+, b^-), i.e. a free $\mathbb{Z}-$ module of finite rank. Here $(\cdot,\cdot)$ is a bilinear form and $q :\ x \mapsto x^2/2 = \frac{1}{2}(x,x)$ the corresponding quadratic form which takes values in $\mathbb{Z}$ on L. By $\operatorname{sig}(L)$ we mean the signature $b^+ - b^-$ of L. We write

$$L' = \left\{x \in L \otimes \mathbb{Q} \quad (x,y) \in \mathbb{Z} \text{ for all } y \in L\right\}$$

for the dual lattice of L. Since L is assumed to be even, $L \subset L'$ and L'/L is a finite abelian group. For the rest of the discussion, let $N \in \mathbb{N}$ be the smallest integer such that $Nx^2/2 \in \mathbb{Z}$ for all $x \in L'$. N is called the level of L. For $n \in \mathbb{N}$ we define the subgroups

$$(L'/L)^n = \left\{\mu \in L'/L;\quad \exists\, \nu \in L'/L :\ \mu = n\nu\right\},$$
$$(L'/L)_n = \left\{\mu \in L'/L;\quad n\mu = 0\right\}$$

and the subset

$$(L'/L)^{n*} = \{\mu \in L'/L; \quad (\mu, \nu) \equiv n\nu^2/2 \pmod 1 \text{ for all } \nu \in (L'/L)_n\}$$

of L'/L.

PROPOSITION 2.1. *$(L'/L)^{n*}$ is a coset of $(L'/L)^n$ in L'/L.*

Proof. See [**28**] or [**16**], Proposition 1.34, for a proof. □

We adopt the following notation for $(L'/L)^{n*}$ from [**28**]. We obviously have

$$(L'/L)^{n*} = \mu_n + (L'/L)^n,$$

where μ_n denotes a coset representative. For an element $\nu = \mu_n + n\mu \in (L'/L)^{n*}$ we define

$$\nu_n^2/2 := n\mu^2/2 + (\mu_n, \mu). \tag{2.1}$$

Proposition 2.2 in [**28**] shows that (2.1) is well defined.

The next proposition gives a criterion when $(L'/L)^{n*}$ is equal to $(L'/L)^n$. This criterion is based on the concept of Jordan blocks which is described e.g. in [**29**] or [**36**].

PROPOSITION 2.2 ([**4**], p. 324, [**28**], p. 6). *$(L'/L)^{n*}$ is equal to $(L'/L)^n$ if and only if the 2-adic Jordan block of type 2^k with $2^k \| n$ is even.*

For the discriminant form $(L'/L, q)$ having the above properties, we introduce the *Weil representation* associated to $(L'/L, q)$. The Weil representation is a representation

$$\rho_L : \widetilde{\Gamma}(1) \longrightarrow \mathrm{GL}(\mathbb{C}[L'/L])$$

of $\widetilde{\Gamma}(1)$ on the group ring $\mathbb{C}[L'/L]$ defined by

$$\rho_L(T)(\mathfrak{e}_\lambda) = e(\lambda^2/2)\mathfrak{e}_\lambda, \tag{2.2}$$

$$\rho_L(S)(\mathfrak{e}_\lambda) = \frac{e(-\operatorname{sig}(L)/8)}{\sqrt{|L'/L|}} \sum_{\mu \in L'/L} e(-(\lambda, \mu))\mathfrak{e}_\mu. \tag{2.3}$$

Note that

$$\rho_L(Z)(\mathfrak{e}_\gamma) = e(-\operatorname{sig}(L)/4)\mathfrak{e}_{-\lambda}. \tag{2.4}$$

Here $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$ and $(\mathfrak{e}_\lambda)_{\lambda \in L'/L}$ denotes the standard basis of $\mathbb{C}[L'/L]$. From (2.4) it follows that Z^2 acts trivially if the signature of L is even. Therefore, ρ_L factors through $\Gamma(1)$ in this case. Moreover, if the signature of L is even, it can be proved that the Weil representation is trivial on the principal congruence subgroup $\Gamma(N)$ (see e.g. [**12**], Chapter 3, Theorem 3.2), where N is the level of L. Therefore, ρ_L factors through the finite group $\Gamma(1)/\Gamma(N) \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. If the signature of L is odd, the Weil representation is trivial on the group

$$\Gamma(N)^* = \left\{\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\frac{c}{d}\right)\sqrt{c\tau + d}\right); \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(N)\right\} \lhd \widetilde{\Gamma}(1),$$

where $\left(\frac{c}{d}\right)$ is the Kronecker symbol, see [**10**], p. 253 for a sketch of a proof. In this case, the Weil representation factors through the finite group $\widetilde{\Gamma}(1)/\Gamma(N)^*$. Denote by $g_n(L)$ the Gauß sum

$$g_n(L) = \sum_{\lambda \in L'/L} e(n\lambda^2/2). \tag{2.5}$$

For $n = 1$, Milgram's formula holds

$$g(L) := g_1(L) = \sqrt{|L'/L|}e(\mathrm{sig}(L)/8) \tag{2.6}$$

(see [**21**], Appendix 4).

We now define vector-valued modular forms of type ρ_L. With respect to the standard basis a function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ can be written in the form

$$f(\tau) = \sum_{\lambda \in L'/L} f_\lambda(\tau)\mathfrak{e}_\lambda.$$

The following operator generalises the usual Petersson slash operator to the space of all those functions: For $k \in \frac{1}{2}\mathbb{Z}$ define

$$f \mid_{k,L} (\gamma, \phi) = \phi(\gamma, \tau)^{-2k} \rho_L(\gamma, \phi)^{-1} f(\gamma\tau). \tag{2.7}$$

A holomorphic function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is called a modular form of weight k and type ρ_L for $\widetilde{\Gamma}(1)$ if $f \mid_{k,L} (\gamma, \phi) = f$ for all $(\gamma, \phi) \in \widetilde{\Gamma}(1)$, and if f is holomorphic at the cusp ∞. The last condition means that every component f_λ of f has a Fourier expansion of the form $f_\lambda(\tau) = \sum_{n \geq 0,\, n \in \mathbb{Z}+\lambda^2/2} c(\lambda, n)e(n\tau)$. We denote by $M_{k,L}$ the space of all such modular forms, by $S_{k,L}$ the subspace cusp forms. For more details see e.g. [**6**] or [**10**]. Note that formula (2.4) implies that $M_{k,L} = \{0\}$ unless

$$2k \equiv \mathrm{sig}(L) \pmod 2. \tag{2.8}$$

Therefore, if the signature of L is even, only non-trivial spaces of integral weight can occur, if the signature of L is odd only non-trivial spaces of half-integral weight can occur. The Petersson scalar product on $S_{k,L}$ is given by

$$(f, g) = \int_{\Gamma(1)\backslash\mathbb{H}} \langle f(\tau), g(\tau)\rangle \,\mathrm{Im}(\tau)^k d\mu(\tau) \tag{2.9}$$

where

$$d\mu(\tau) = \frac{dx\, dy}{y^2}$$

denotes the hyperbolic volume element and

$$\left\langle \sum_{\lambda \in L'/L} a_\lambda \mathfrak{e}_\lambda, \sum_{\lambda \in L'/L} b_\lambda \mathfrak{e}_\lambda \right\rangle = \sum_{\lambda \in L'/L} a_\lambda \overline{b_\lambda} \tag{2.10}$$

is the standard scalar product on the group ring $\mathbb{C}[L'/L]$. For $\lambda, \mu \in L'/L$ and $(\gamma, \phi) \in \widetilde{\Gamma}(1)$ we define the coefficient $\rho_{\lambda\mu}(M, \phi)$ of the representation ρ_L by

$$\rho_{\lambda\mu}(\gamma, \phi) = \langle \rho_L(\gamma, \phi)\mathfrak{e}_\mu, \mathfrak{e}_\lambda\rangle. \tag{2.11}$$

3. Computation of the character of the Weil representation. The goal of this section is to present an explicit formula for the expression

$$\operatorname{tr}(\rho_L(\tilde{\gamma})) = \sum_{\mu \in L'/L} \langle \rho_L(\tilde{\gamma})\mathfrak{e}_\mu, \mathfrak{e}_\mu \rangle \tag{3.1}$$

for $\gamma \in \Gamma(1)$. It will turn out that the character of the Weil representation (3.1) occurs as a part of the trace of the Hecke operator $T(n)$. Our formula is based on an explicit formula for the Weil representation given by Strömberg in [**36**], p. 510 and Theorem 6.4:

$$\rho_L(\gamma, j(\gamma,\tau))\mathfrak{e}_\mu = \xi(\gamma)\sqrt{|(L'/L)^c|/|L'/L|} \sum_{\nu \in (L'/L)^{c*}} e(a\nu_c^2/2 + bd\mu^2/2 + b(\mu,\nu))\mathfrak{e}_{\nu+d\mu}, \tag{3.2}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $\Gamma(1)$.

The constant $\xi(\gamma)$ is specified in the cited theorem. This formula is valid for odd and even signature of the lattice L which implies the same for the formula of the character.

Theorem 3.1. *Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$ and $\mu_c \in L'/L$ be a coset representative of $(L'/L)^{c*}$ in L'/L. Then the character of $\rho_L((\gamma, j(\gamma,\tau))$ can be written as*

$$\xi(\gamma)\sqrt{|(L'/L)_c|/|L'/L|}\, e\Big(-\frac{a}{c}\mu_c^2/2\Big) \sum_{\substack{\lambda \in L'/L, \\ (1-d)\lambda - \mu_c \in (L'/L)^c}} e\Big(\frac{1}{c}(a+d-2)\lambda^2/2\Big) \tag{3.3}$$

if $c \neq 0$ and as

$$e((-\operatorname{sig}(L)/8)(1 - \operatorname{sgn}(d)) \sum_{\substack{\mu \in L'/L, \\ \mu = a\mu}} e(ab\mu^2/2) \tag{3.4}$$

if $c = 0$.

Proof. The character of $\rho_L(\gamma, j(\gamma,\tau))$ is given by (3.1). Since for every $\lambda \in L'/L$ and every $\nu = \mu_c + \mu' \in (L'/L)^{c*}$ the equation $\mu_c + \mu' + d\lambda = \lambda$ is fulfilled for all $\mu' \in (L'/L)^c$ with $\mu' = (1-d)\lambda - \mu_c$ we obtain with the help of (2.1) and (3.2)

$$\begin{aligned}
&\operatorname{tr}(\rho_L(\tilde{\gamma})) \\
&= \xi(\gamma)\sqrt{|(L'/L)_c|/|L'/L|} \sum_{\lambda \in L'/L} \langle \sum_{\nu \in (L'/L)^{c*}} e(b(\nu,\lambda) + bd\lambda^2/2) e(a\nu_c^2/2)\mathfrak{e}_{\nu+d\lambda}, \mathfrak{e}_\lambda \rangle \\
&= \xi(\gamma)\sqrt{|(L'/L)_c|/|L'/L|} \\
&\quad \times \sum_{\substack{\lambda \in L'/L \\ (1-d)\lambda - \mu_c \in (L'/L)^c}} e\big(bd\lambda^2/2 + b((1-d)\lambda, \lambda) + ac(((1-d)\lambda - \mu_c)/c)^2/2 \\
&\quad + a(((1-d)\lambda - \mu_c)/c, \mu_c)\big) \\
&= \xi(\gamma)\sqrt{|(L'/L)_c|/|L'/L|} \sum_{\substack{\lambda \in L'/L, \\ (1-d)\lambda - \mu_c \in (L'/L)^c}} e(2b\lambda^2/2 - bd\lambda^2/2 \\
&\qquad + \frac{a}{c}(((1-d)\lambda)^2/2 - \mu_c^2/2)).
\end{aligned}$$

The last equation can be verified by a straightforward computation. Since

$$e(2b\lambda^2/2 - bd\lambda^2/2 + \frac{a}{c}((1-d)\lambda)^2/2) = e\Big(\frac{1}{c}\lambda^2/2\,(a + d(ad-bc) - 2(ad-bc))\Big)$$

the last expression above for the trace can be written as

$$\xi(\gamma)\sqrt{|(L'/L)_c|/|L'/L|}\,e\Big(-\frac{a}{c}\mu_c^2/2\Big) \sum_{\substack{\lambda \in L'/L \\ (1-d)\lambda - \mu_c \in (L'/L)^c}} e\Big(\frac{1}{c}(a+d-2)\lambda^2/2\Big).$$

The formula for the character for the case $c = 0$ follows immediately from Shintani's formula, see e.g. [**6**], (1.5). □

REMARK 3.2. Note that the formula (3.3) for the character of the Weil representation is independent of the choice of μ_c.

The following theorem provides a very explicit formula for the character of the Weil representation for the special case of $|L'/L| = p$ where p is an odd prime p. In this case it can be shown that the level of L is p (see [**8**], p. 50).

THEOREM 3.3. *Let p be an odd prime and $|L'/L| = p$. Then we have for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma(1)$*

$$\operatorname{tr}(\rho_L(\gamma)) = \begin{cases} \left(\frac{a+d-2}{p}\right), & \text{if } 2 \not\equiv a+d \pmod{p}, \\ e(\operatorname{sig}(L)/8)\left(\frac{-c}{p}\right)\sqrt{p}, & \text{if } 2 \equiv a+d \pmod{p} \text{ and } (c,p) = 1, \\ e(\operatorname{sig}(L)/8)\left(\frac{b}{p}\right)\sqrt{p}, & \text{if } 2 \equiv a+d \pmod{p} \text{ and } b \not\equiv c \equiv 0 \pmod{p}, \\ p, & \text{if } 2 \equiv a+d \pmod{p} \text{ and } b \equiv c \equiv 0 \pmod{p}. \end{cases} \tag{3.5}$$

Proof. The proof is based on the formula (3.3) and the two formulae in [**28**], Proposition 4.8, which allow an explicit description of the expression $\xi(\gamma)$ in (3.3) for the cases $(c, p) = 1$ and $p \mid c$. To further simplify Scheithauer's formulas in [**28**] we use the fact that $\operatorname{oddity}(L'/L) \equiv 0 \bmod 8$ if the level of L is odd, see [**38**], Lemma 5.8, and Theorem 1.5.2 in [**2**]. □

REMARK 3.4. The formula (3.5) can also be found in [**37**], Theorem 1A, p. 222 and the example on p. 224. However, there it is proved in a different way.

4. Hecke operators on vector-valued modular forms. In this chapter, we briefly recall how Hecke operators on vector-valued modular forms can be defined. All details can be found in [**10**]. Note that in this paper, we consider only Hecke operators $T(n)$, where n is *coprime* to the level N of L. In this case the Hecke operators can be defined as usual by the action of a suitable Hecke algebra.

4.1. The case of even signature. In order to define Hecke operators on $M_{k,L}$ one has to extend the Petersson slash operator in (2.7) to some suitable group which is isomorphic to a subgroup of $\mathrm{GL}_2^+(\mathbb{Q})$. In particular, this means that the Weil representation has to be extended to this group. As a starting point the Weil representation, viewed as a representation of the finite group $S(N) := \Gamma(1)/\Gamma(N)$,

$$\rho_L(A)\mathfrak{e}_\lambda = \rho_L(s(A))\mathfrak{e}_\lambda, \quad A \in S(N), \tag{4.1}$$

can be extended to a group isomorphic to a subgroup of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Here $s : S(N) \to \Gamma(1)$ is a section, that is $\pi_N \circ s = \mathrm{id}_{S(N)}$, where π_N denotes the component-wise reduction modulo N. Let $Q(N)$ be the group

$$Q(N) = \{(M, r) \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \times U(N); \quad \det(M) \equiv r^2 \pmod N\}$$

with the product defined component-wise. Here $U(N) = (\mathbb{Z}/N\mathbb{Z})^*$. By $M \mapsto (M, 1)$ the group $S(N)$ can be embedded into $Q(N)$. Moreover, for $(M, r) \in Q(N)$ the map $(M, r) \mapsto (M \left(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right)^{-1}, r)$ defines an isomorphism $Q(N) \cong S(N) \times U(N)$. Then the Weil representation of $U(N)$ can be defined as follows

$$\rho_L\left(\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, r\right)\mathfrak{e}_\lambda = \frac{g_1(L)}{g_r(L)}\mathfrak{e}_\lambda \tag{4.2}$$

and on the whole group $Q(N)$ by

$$\rho_L(M, r)\mathfrak{e}_\lambda = \rho_L(M \left(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right)^{-1}, 1) \circ \rho_L(\left(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right), r)\mathfrak{e}_\lambda. \tag{4.3}$$

Since the assignment $r \mapsto \frac{g_1(L)}{g_r(L)}$ defines a character of $U(N)$ (see [**10**], p. 256) we obtain that (4.2) is indeed a representation. It is easily seen that (4.3) extends the Weil representation to a representation of $Q(N)$. Consider the groups

$$\begin{aligned} \mathcal{G}(N) = \big\{M \in \mathrm{GL}_2^+(\mathbb{Q}); \quad &\exists n \in \mathbb{Z} \text{ with } (n, N) = 1 \text{ such that } nM \in M_2(\mathbb{Z}) \\ &\text{and } (\det(nM), N) = 1\big\} \end{aligned} \tag{4.4}$$

and

$$\mathcal{Q}(N) = \big\{(M, r) \in \mathcal{G}(N) \times (\mathbb{Z}/N\mathbb{Z})^*; \quad \det(M) \equiv r^2 \pmod N\big\}. \tag{4.5}$$

The modular group $\Gamma(1)$ can be embedded into $\mathcal{Q}(N)$ by $\gamma \mapsto (\gamma, 1)$. The component-wise reduction π_N maps the group $\mathcal{Q}(N)$ into the group $Q(N)$. Therefore, the Weil representation can be extended to the group $\mathcal{Q}(N)$ by

$$\rho_L : \mathcal{Q}(N) \longrightarrow \mathrm{GL}(\mathbb{C}[L'/L]), \quad (M, r) \mapsto \rho_L(\pi_N(M), r), \tag{4.6}$$

where ρ_L on $Q(N)$ is defined by (4.3). The action of $\mathcal{Q}(N)$ on vector-valued functions is then given by

$$f \mid_{k,L} (M, r) = \sum_{\lambda \in L'/L} (f_\lambda \mid_k M)\rho_L^{-1}(M, r)\mathfrak{e}_\lambda \tag{4.7}$$

for $f = \sum_{\lambda \in L'/L} f_\lambda \mathfrak{e}_\lambda$, where

$$f \mid_k M = \det(M)^{k/2} j(M, \tau)^{-2k} f(M\tau) \tag{4.8}$$

is the usual Petersson slash operator. It is easily seen that (4.7) extends the action (2.7) of $\Gamma(1)$ to the group $\mathcal{Q}(N)$.

4.2. The case of odd signature. If the signature of L is odd, there are only non-trivial vector-valued modular forms of half-integral weight. In this case (4.6) and (4.8) define only projective actions of the group $\mathcal{Q}(N)$. In order to obtain honest actions one needs to introduce appropriate central extensions of $\mathcal{Q}(N)$. First, we consider the action on $\mathbb{C}[L'/L]$. By [**10**], p. 253 and p. 259ff. (4.6) yields only a projective representation determined up to a factor ± 1, that is

$$\rho_L : \mathcal{Q}(N) \longrightarrow \mathrm{GL}(\mathbb{C}[L'/L])/\{\pm 1\}, \quad g \mapsto \rho_L(g).$$

Choosing a section $s : \mathrm{GL}(\mathbb{C}[L'/L])/\{\pm 1\} \to \mathrm{GL}(\mathbb{C}[L'/L])$ gives rise to a cocycle $c : \mathcal{Q}(N) \times \mathcal{Q}(N) \to \{\pm 1\}$ and a central group extension

$$\mathcal{Q}_1(N) = \mathcal{Q}(N) \times \{\pm 1\}.$$

By setting

$$\rho_L(M, r, t) = t\rho_L(M, r)$$

for $(M, r, t) \in \mathcal{Q}_1(N)$ we obtain a representation of $\mathcal{Q}_1(N)$. For $(\gamma, 1) \in \Gamma(1) \times \{1\} \subset \mathcal{Q}(N)$ we set

$$\rho_L(\gamma, 1) = \rho_L(\gamma, j(\gamma, \tau)). \tag{4.9}$$

This choice of s yields an injective homomorphism of $\widetilde{\Gamma}(1)$ into $\mathcal{Q}_1(N)$

$$\widetilde{\Gamma}(1) \longrightarrow \mathcal{Q}_1(N), \quad (\gamma, \pm j(\gamma, \tau)) \mapsto (\gamma, 1, \pm 1). \tag{4.10}$$

Moreover, we set

$$\rho_L\left(\begin{pmatrix} m^2 & 0 \\ 0 & 1 \end{pmatrix}, m\right) \mathfrak{e}_\lambda = \mathfrak{e}_{m^{-1}\lambda}. \tag{4.11}$$

On the other hand, if the weight $k \in \mathbb{Z} + \frac{1}{2}$ the action (4.8) defines a cocyle, which is determined by the square root of the automorphic factor. One can show that this cocyle is not isomorphic to the cocycle c on the group $\mathcal{Q}(N)$. However, their restriction to $\widetilde{\Gamma}(1)$ are isomorphic (compare with [**10**], p. 260 and with [**5**], Chapter IV, Section 1–3, for the theory of group extensions and group cohomology used before).

Therefore, in order to define an action on vector-valued functions one has to define a central group extension

$$\begin{aligned} \mathcal{Q}_2(N) = \big\{&(M, \phi(M, \tau), r, t); \quad M \in \mathcal{G}(N),\ r \in (\mathbb{Z}/N\mathbb{Z})^*, \\ &\det(M) \equiv r^2 \bmod N,\ t \in \{\pm 1\}\big\} \end{aligned} \tag{4.12}$$

of $\mathcal{Q}_1(N)$ by $\{\pm 1\}$. Because of (4.10) there is an injective homomorphism

$$\mathcal{L} : \widetilde{\Gamma}(1) \longrightarrow \mathcal{Q}_2(N), \quad (\gamma, \pm j(\gamma, \tau)) \mapsto (\gamma, \pm j(\gamma, \tau), 1, \pm 1). \tag{4.13}$$

For an element $\widetilde{M} = (M, \phi(M, \tau), r, t) \in \mathcal{Q}_2(N)$, we set

$$\rho_L(M, \phi(M, \tau), r, t) = \rho_L(M, r, t),$$

that is, we compose the projection to the group $\mathcal{Q}_1(N)$ with the Weil representation on that group. By the definition of the embedding $\mathcal{L}$, we have

$$\rho_L(\mathcal{L}(\gamma)) = \rho_L(\gamma) \tag{4.14}$$

for $\gamma \in \widetilde{\Gamma}(1)$. Note that the Weil representation on $\mathcal{Q}(N)$ and $\mathcal{Q}_2(N)$ is unitary with respect to the scalar product (2.10). The action of $\mathcal{Q}_2(N)$ on vector-valued functions $f = \sum_{\lambda \in L'/L} f_\lambda \mathfrak{e}_\lambda$ is given by

$$f \mid_{k,L} (M, \phi(M,\tau), r, t) = \sum_{\lambda \in L'/L} (f_\lambda \mid_k (M,\phi))\rho_L^{-1}(M, r, t)\mathfrak{e}_\lambda, \tag{4.15}$$

where $f \mid_k (M,\phi) = \phi(M,\tau)^{-2k} f(M\tau)$.

4.3. Hecke operators. Let $n \in \mathbb{N}$ be *coprime* to the level N of L. The Hecke operator $T(n)$ will be defined in terms of the action of the Hecke algebra given by the pair of groups $(\mathcal{Q}(N), \Gamma(1) \times \{1\})$ or $(\mathcal{Q}_2(N), \mathcal{L}(\widetilde{\Gamma}(1)))$ depending on the parity of the signature of the lattice L. The definition of the group $\mathcal{Q}(N)$ implies that we have to assume that n is a square modulo N. Further, if the signature of L is odd one can show that $T(n)$ is zero unless n is a square in $\mathbb{Z}$, see [**10**], Proposition 4.9. Therefore, we assume in the following section that $(n, N) = 1$ and

$$\begin{aligned} &n \equiv r^2 \bmod N, \ \text{if } \operatorname{sig}(L) \text{ is even}, \\ &n = m^2, \ \text{if } \operatorname{sig}(L) \text{ is odd}. \end{aligned} \tag{4.16}$$

In order to keep the notation as simple as possible and to treat the cases of odd and even signature in the following section simultaneously, we introduce the following notation:

$$\Gamma = \begin{cases} \Gamma(1) \times \{1\} \subset \Gamma(1) \times (\mathbb{Z}/N\mathbb{Z})^*, & \text{if } \operatorname{sig}(L) \text{ even}, \\ \mathcal{L}(\widetilde{\Gamma}(1)), & \text{if } \operatorname{sig}(L) \text{ odd}, \end{cases} \tag{4.17}$$

where $\mathcal{L}$ denotes the embedding (4.13) and accordingly

$$\mathcal{M}(n) = \begin{cases} \Gamma(\left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right), r)\Gamma, & \text{if } \operatorname{sig}(L) \text{ even}, \\ \Gamma(\left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right), 1, m, 1)\Gamma, & \text{if } \operatorname{sig}(L) \text{ odd}. \end{cases} \tag{4.18}$$

As usual, the Hecke operator is then defined by

$$f \mid_{k,L} T(n) = n^{k/2-1} \sum_{\widetilde{M} \in \Gamma \backslash \mathcal{M}(n)} f \mid_{k,L} \widetilde{M}, \tag{4.19}$$

where the slash operator is given by (4.7) and (4.15) depending on the parity of the signature of L. The following theorem is proved in [**10**], Theorem 4.12, p. 266.

THEOREM 4.1. *Let $m, n \in \mathbb{N}$ be coprime and defined as in* (4.16)*. Then the Hecke operator $T(n)$ is a linear operator on $M_{k,L}$ taking cusp forms to cusp forms. It is self-adjoint with respect to the Petersson scalar product* (2.9)*. Moreover,*

$$T(m)T(n) = T(mn).$$

5. A kernel function. In the following, we define a vector-valued function which turns out to be a kernel function of the Hecke operator $T(n)$.

DEFINITION 5.1. Let $n \in \mathbb{N}$ be as in (4.16) and $k \in \frac{1}{2}\mathbb{Z}$ with $k > 2$. Moreover, let $\beta \in L'$ and $\tau, \tau' \in \mathbb{H}$. Then $h_{\beta,n} : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{C}[L'/L]$ is defined by

$$\begin{aligned} h_{\beta,n}(\tau, \tau') &= \sum_{\widetilde{M} \in \mathcal{M}(n)} (\tau + \tau')^{-k} \mathfrak{e}_\beta \mid_{k,L} \widetilde{M} \\ &= \sum_{\widetilde{M} \in \mathcal{M}(n)} \phi(M, \tau)^{-2k} (M\tau + \tau')^{-k} \rho_L^{-1}(\widetilde{M}) \mathfrak{e}_\beta, \end{aligned} \tag{5.1}$$

where the slash-action is in the variable τ and $M \in \mathcal{G}(N)$ is the projection of $\widetilde{M}$ onto the first component.

We extend the definition (2.11) for the coefficient of the representation to elements $\widetilde{M} \in \mathcal{M}(n)$ by setting

$$\rho_{\delta\beta}(\widetilde{M}) = \langle \rho_L(\widetilde{M}) \mathfrak{e}_\beta, \mathfrak{e}_\delta \rangle.$$

Then each component function of $h_{\beta,n}$ can be written as

$$\sum_{\widetilde{M} \in \mathcal{M}(n)} \rho_{\delta\beta}(\widetilde{M}^{-1}) \phi(M, \tau)^{-2k} (M\tau + \tau')^{-k}. \tag{5.2}$$

Since the representation ρ_L factors through a finite group and $k > 2$ one can show as in the scalar-valued case (see [**39**], p. 45 and [**23**], Theorem 1, p. 10) that for each component δ the series (5.2) converges normally on $\mathbb{H} \times \mathbb{H}$ and is holomorphic in each variable. Therefore, $h_{\beta,n}$ is a holomorphic function in each variable. By Remark 5.2 (*a*) below and the absolute convergence a standard argument proves that $h_{\beta,n}$ is invariant under the slash action $|_{k,L}$ of Γ with respect to the variable τ. Moreover, for all $\tau' \in \mathbb{H}$

$$\sum_{\widetilde{M} \in \mathcal{M}(n)} \phi(M, \tau)^{-2k} (M\tau + \tau')^{-k}$$

is a cusp form of the variable τ (see [**23**], Théorème 1, p. 10) which implies that $h_{\beta,n}(\cdot, \tau')$ is holomorphic at ∞. This means that $h_{\beta,n}$ is a vector-valued cusp form with respect to τ.

REMARK 5.2. Let $n, n' \in \mathbb{N}$ be as in (4.16) and $\widetilde{A} \in \mathcal{M}(n')$. Then the following equations hold:

(a)

$$h_{\beta,n} \mid_{k,L} \widetilde{A} = \sum_{\widetilde{M} \in \mathcal{M}(n)} \left((\tau + \tau')^{-k} \mathfrak{e}_\beta \mid_{k,L} \widetilde{M} \mid_{k,L} \widetilde{A} \right).$$

(b)

$$h_{\beta,1} \mid_{k,L} T(n) = n^{k/2-1} h_{\beta,n}.$$

Both the slash operator and the Hecke operator are applied with respect to the first variable τ.

Proof. (a) The first statement is essentially about interchanging the sum over $\mathcal{M}(n)$ and the slash operator, which is allowed since (5.2) converges normally.

(b) For the second statement note that by the definition of the Hecke operator and (a) we have

$$\begin{aligned} h_{\beta,1} \mid_{k,L} T(n) &= n^{k/2-1} \sum_{\widetilde{A}\in\Gamma\backslash\mathcal{M}(n)} h_{\beta,1} \mid_{k,L} \widetilde{A} \\ &= n^{k/2-1} \sum_{\widetilde{A}\in\Gamma\backslash\mathcal{M}(n)} \sum_{\gamma\in\Gamma} \left((\tau+\tau')^{-k}\mathfrak{e}_\beta \mid_{k,L} \gamma \mid_{k,L} \widetilde{A}\right) \\ &= n^{k/2-1} \sum_{\widetilde{M}\in\mathcal{M}(n)} (\tau+\tau')^{-k}\mathfrak{e}_\beta \mid_{k,L} \widetilde{M}. \end{aligned}$$

□

We now define an integral operator on the space of vector-valued cusp forms.

DEFINITION 5.3. Let $k \in \frac{1}{2}\mathbb{Z}$ with $k > 2$ and $n \in \mathbb{N}$ as in (4.16). For $f \in S_{k,L}$ let

$$(K_n f)(\tau') = \sum_{\beta\in L'/L} \left(\int_{\Gamma(1)\backslash\mathbb{H}} \langle f(\tau), h_{\beta,n}(\tau, -\overline{\tau'})\rangle \operatorname{Im}(\tau)^k d\mu(\tau)\right)\mathfrak{e}_\beta. \tag{5.3}$$

Note that the operator K_n is well defined on $S_{k,L}$ since the integral is the Petersson scalar product defined in (2.9).

LEMMA 5.4 (See [**23**], p. 42 or [**39**], p. 46). *Let $k > 2$ be a real number and $f : \mathbb{H} \to \mathbb{C}$ be holomorphic such that $f(\tau)(\operatorname{Im}(\tau))^{k/2}$ is bounded on $\mathbb{H}$. Then for all $\tau' \in \mathbb{H}$*

$$\int_{\mathbb{H}} (x - iy - \tau')^{-k} f(x+iy) y^{k-2} dx\, dy = \frac{\overline{C_k}}{2} f(\tau'),$$

where

$$C_k = \frac{\pi}{k-1} i^{-k} 2^{3-k}. \tag{5.4}$$

REMARK 5.5. Note that the constant C_k differs by a factor $1/2$ from the corresponding constant in [**23**], p. 10.

THEOREM 5.6. *Let $k \in \frac{1}{2}\mathbb{Z}$, $k > 2$, and $f = \sum_{\lambda\in L'/L} f_\lambda \mathfrak{e}_\lambda \in S_{k,L}$. Then, we have*

$$(K_1 f)(\tau') = \overline{C_k} f(\tau')$$

for all $\tau' \in \mathbb{H}$, where C_k is defined in Lemma 5.4.

Proof. The proof is essentially the same as the proof of [**39**], Theorem 1(i), pp. 45–46. It suffices to prove that

$$\int_{\Gamma(1)\backslash\mathbb{H}} \langle f(\tau), h_{\beta,1}(\tau, -\overline{\tau'})\rangle \operatorname{Im}(\tau)^k d\mu(\tau) = \overline{C_k} f_\beta(\tau') \tag{5.5}$$

for all $\beta \in L'/L$. Let $\mathcal{F} = \Gamma(1)\backslash\mathbb{H}$ be a fundamental domain for $\Gamma(1)$. Since ρ_L is unitary with respect to the scalar product (2.10), we obtain that

$$\begin{aligned}\left\langle f(\tau), \phi(\gamma,\tau)^{-2k}\rho_L^{-1}(\gamma)\ (\gamma\tau-\overline{\tau'})^{-k}\mathfrak{e}_\beta\right\rangle \operatorname{Im}(\tau)^k \\ = \left\langle \phi(\gamma,\tau)^{2k}\rho_L(\gamma)f(\tau), (\gamma\tau-\overline{\tau'})^{-k}\mathfrak{e}_\beta\right\rangle \operatorname{Im}(\gamma\tau)^k\end{aligned} \tag{5.6}$$

for all $\gamma \in \Gamma$. Therefore, since f is a vector-valued modular form, we get

$$\begin{aligned}&\int_{\mathcal{F}}\left\langle f(\tau), \sum_{\gamma\in\Gamma}(\tau-\overline{\tau'})^{-k}\mathfrak{e}_\beta\mid_{k,L}\gamma\right\rangle \operatorname{Im}(\tau)^k d\mu(\tau)\\ &= \int_{\mathcal{F}}\sum_{\gamma\in\Gamma}\left\langle f(\tau), (\tau-\overline{\tau'})^{-k}\mathfrak{e}_\beta\mid_{k,L}\gamma\right\rangle \operatorname{Im}(\tau)^k d\mu(\tau)\\ &\underset{(5.6)}{=} \int_{\mathcal{F}}\sum_{\gamma\in\Gamma}\left\langle f(\gamma\tau), (\gamma\tau-\overline{\tau'})^{-k}\mathfrak{e}_\beta\right\rangle \operatorname{Im}(\gamma\tau)^k d\mu(\tau)\\ &= \sum_{\gamma\in\Gamma}\int_{\gamma\mathcal{F}}\left\langle f(\tau), (\tau-\overline{\tau'})^{-k}\mathfrak{e}_\beta\right\rangle \operatorname{Im}(\tau)^k d\mu(\tau).\end{aligned}$$

Using the same arguments as in [**39**], p. 46 below formula (9), we obtain that the last expression equals

$$2\int_{\mathbb{H}} f_\beta(x+iy)(x-iy-\tau')^{-k}y^k\frac{dx\,dy}{y^2}.$$

The result now follows from Lemma 5.4. □

THEOREM 5.7. *Let* $k \in \frac{1}{2}\mathbb{Z}$, $k > 2$ *and* $n \in \mathbb{N}$ *as in* (4.16). *Then for all* $f \in S_{k,L}$ *the following equation holds for all* $\tau' \in \mathbb{H}$

$$(K_n f)(\tau') = n^{-k/2+1}\overline{C_k}(f\mid_{k,L}T(n))(\tau'), \tag{5.7}$$

where C_k *is defined by* (5.4).

Proof. By Remark 5.2 (b) and the fact that the Hecke operator is self-adjoint with respect to the Petersson scalar product (2.9) we have

$$\begin{aligned}(K_n f)(\tau') &= \sum_{\beta\in L'/L}\left(\int_{\Gamma(1)\backslash\mathbb{H}}\langle f(\tau), h_{\beta,n}(\tau,-\overline{\tau'})\rangle \operatorname{Im}(\tau)^k d\mu(\tau)\right)\mathfrak{e}_\beta\\ &= n^{-k/2+1}\sum_{\beta\in L'/L}\left(\int_{\Gamma(1)\backslash\mathbb{H}}\langle f(\tau), h_{\beta,1}(\tau,-\overline{\tau'})\mid_{k,L}T(n)\rangle \operatorname{Im}(\tau)^k d\mu(\tau)\right)\mathfrak{e}_\beta\\ &= n^{-k/2+1}\sum_{\beta\in L'/L}\left(\int_{\Gamma(1)\backslash\mathbb{H}}\langle f(\tau)\mid_{k,L}T(n), h_{\beta,1}(\tau,-\overline{\tau'})\rangle \operatorname{Im}(\tau)^k d\mu(\tau)\right)\mathfrak{e}_\beta.\end{aligned} \tag{5.8}$$

The statement of the theorem now follows from Theorem 5.6. □

For the next theorem, we use the following notation: By $(f_i)_{i=1,\dots,d}$ we denote a basis of simultaneous eigenforms of all Hecke operators $T(n)$ in $S_{k,L}$, i.e. $f_i\mid_{k,L}T(n) = \lambda(n)^i f_i$ for all $i = 1,\dots,d$. Note that this basis is chosen to be orthogonal with respect to the

Petersson scalar product (2.9). Furthermore, we assume that the basis is normalised (i.e. $(f_i, f_i) = 1$) and write

$$f_i = \sum_{\lambda \in L'/L} f_\lambda^i \mathfrak{e}_\lambda.$$

THEOREM 5.8. *Let $n \in \mathbb{N}$ be as in* (4.16). *Then the following statements hold:*
(*a*)

$$h_{\beta,n}(\tau, -\overline{\tau'}) = n^{-k/2+1} C_k \sum_{i=1}^{d} \lambda^i(n) \overline{f_\beta^i(\tau')} f_i(\tau). \tag{5.9}$$

(*b*)

$$\operatorname{tr} T(n) = C_k^{-1} n^{k/2-1} \sum_{\beta \in L'/L} \int_{\Gamma(1)\backslash \mathbb{H}} \langle h_{\beta,n}(\tau, -\overline{\tau}), \mathfrak{e}_\beta \rangle \operatorname{Im}(\tau)^k d\mu(\tau). \tag{5.10}$$

Proof. (a) Since $h_{\beta,n}(\cdot, \tau')$ is a cusp form for every $\tau' \in \mathbb{H}$ we can use the orthogonal basis $(f_i)_{i=1,\dots,d}$ to write

$$\begin{aligned} h_{\beta,n}(\tau, -\overline{\tau'}) &= \sum_{i=1}^{d} \left(h_{\beta,n}(\cdot, -\overline{\tau'}), f_i\right) f_i(\tau) \\ &= \sum_{i=1}^{d} \overline{\left(f_i, h_{\beta,n}(\cdot, -\overline{\tau'})\right)} f_i(\tau). \end{aligned} \tag{5.11}$$

With the help of (5.5) and (5.8) the right-hand side of (5.11) can be written in the form

$$n^{-k/2+1} \sum_{i=1}^{d} \overline{\left(f_i \mid_{k,L} T(n), h_{\beta,1}(\cdot, -\overline{\tau'})\right)} f_i(\tau) = n^{-k/2+1} C_k \sum_{i=1}^{d} \lambda^i(n) \overline{f_\beta^i(\tau')} f_i(\tau).$$

The last equation holds since the eigenvalues $\lambda^i(n)$ are real.

(b) Replacing $h_{\beta,n}$ with the right-hand side of (5.9) and using the normalisation of the basis $(f_i)_{i=1,\dots,d}$ we obtain

$$\begin{aligned} &C_k^{-1} n^{k/2-1} \sum_{\beta \in L'/L} \int_{\Gamma(1)\backslash \mathbb{H}} \langle h_{\beta,n}(\tau, -\overline{\tau}), \mathfrak{e}_\beta \rangle \operatorname{Im}(\tau)^k d\mu(\tau) \\ &= \sum_{i=1}^{d} \lambda^i(n) \int_{\Gamma(1)\backslash \mathbb{H}} \left\langle f_i, \sum_{\beta \in L'/L} f_\beta^i \mathfrak{e}_\beta \right\rangle \operatorname{Im}(\tau)^k d\mu(\tau) \\ &= \operatorname{tr} T(n). \end{aligned}$$

□

6. Computation of the trace of the Hecke operator. In this section an explicit formula for the trace of the Hecke operator $T(n)$ is computed. As a corollary a dimension formula for the space of cusp forms $S_{k,L}$ is presented. In contrast to

Sections 3 and 5, we now assume that the signature of L is even. Therefore, only vector-valued modular forms of integral weight are considered. By abuse of notation, we will frequently write $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ as well for an element of $\mathcal{M}(n)$ or Γ (see (4.17)), respectively, as for an element of $\mathrm{SL}_2(\mathbb{R})$ and remark that we identify $\Gamma = \Gamma(1) \times \{1\}$ with $\Gamma(1)$ by $(M, 1) \mapsto M$. Moreover, conjugating an element of $\mathcal{M}(n)$ with respect to Γ is equivalent to conjugating the first component of this element with respect to Γ. Thus in order to determine the Γ-conjugacy classes of elements of $\mathcal{M}(n)$ it suffices to determine the $\Gamma(1)$-conjugacy classes of the corresponding matrices.

LEMMA 6.1. *Let $M \in \mathcal{M}(n)$ and $\gamma \in \Gamma$. Then*

$$\sum_{\beta \in L'/L} \langle \rho_L(\gamma^{-1} M \gamma)\mathfrak{e}_\beta, \mathfrak{e}_\beta\rangle = \sum_{\delta \in L'/L} \langle \rho_L(M)\mathfrak{e}_\delta, \mathfrak{e}_\delta\rangle. \tag{6.1}$$

Proof. Since ρ_L is unitary with respect to $\langle \cdot, \cdot\rangle$ we have

$$\begin{aligned}
\langle \rho_L(\gamma^{-1} M \gamma)\mathfrak{e}_\beta, \mathfrak{e}_\beta\rangle &= \langle \rho_L(M)\rho_L(\gamma)\mathfrak{e}_\beta, \rho_L(\gamma)\mathfrak{e}_\beta\rangle \\
&= \left\langle \rho_L(M)\left(\sum_{\delta \in L'/L} \rho_{\delta\beta}(\gamma)\mathfrak{e}_\delta\right), \sum_{\delta' \in L'/L} \rho_{\delta'\beta}(\gamma)\mathfrak{e}_{\delta'}\right\rangle \\
&= \sum_{\delta, \delta' \in L'/L} \rho_{\delta\beta}(\gamma)\overline{\rho_{\delta'\beta}(\gamma)}\langle \rho_L(M)\mathfrak{e}_\delta, \mathfrak{e}_{\delta'}\rangle.
\end{aligned}$$

Therefore,

$$\sum_{\beta \in L'/L} \langle \rho_L(\gamma^{-1} M \gamma)\mathfrak{e}_\beta, \mathfrak{e}_\beta\rangle = \sum_{\delta, \delta' \in L'/L} \langle \rho_L(M)\mathfrak{e}_\delta, \mathfrak{e}_{\delta'}\rangle \sum_{\beta \in L'/L} \rho_{\delta\beta}(\gamma)\overline{\rho_{\delta'\beta}(\gamma)}.$$

By the orthogonality relations of the coefficients $\rho_L(\gamma)$

$$\sum_{\beta \in L'/L} \rho_{\delta\beta}(\gamma)\overline{\rho_{\delta'\beta}(\gamma)} = \begin{cases} 1 & \text{if } \delta = \delta', \\ 0 & \text{otherwise,}\end{cases}$$

we obtain the result. □

The following lemma provides a formula for the Weil representation for special triangular matrices.

LEMMA 6.2. *Let N be the level of L and choose $n \in \mathbb{N}$, $(n, N) = 1$, as in (4.16). Moreover, let $t, r \in \mathbb{Z}$ with $(t, r) = 1$ and $t^2 = 4n$. Then $\left(\left(\begin{smallmatrix} t/2 & r \\ 0 & t/2\end{smallmatrix}\right), t/2\right) \in \mathcal{M}(n)$ and*

$$\rho_L^{-1}\left(\left(\begin{smallmatrix} t/2 & r \\ 0 & t/2\end{smallmatrix}\right), t/2\right)\mathfrak{e}_\mu = \frac{g_{t/2}(L)}{g(L)}\, e\big(-r(t/2)^{-1}\mu^2/2\big)\mathfrak{e}_\mu, \tag{6.2}$$

where $(t/2)^{-1}$ denotes the inverse of $t/2$ in $(\mathbb{Z}/N\mathbb{Z})^*$.

Proof. This is clear because of the equation

$$\rho_L^{-1}\left(\left(\begin{smallmatrix} t/2 & r \\ 0 & t/2\end{smallmatrix}\right), t/2\right) = \rho_L\left(\left(\begin{smallmatrix} 1 & -r(t/2)^{-1} \\ 0 & 1\end{smallmatrix}\right), 1\right)\rho_L^{-1}\left(\left(\begin{smallmatrix} t/2 & 0 \\ 0 & t/2\end{smallmatrix}\right), t/2\right)$$

and the definitions (4.2), (4.3) and (4.6) of the Weil representation on $\mathcal{Q}(N)$. □

In the following, we derive an explicit formula for the integral (5.10). It is the main result of this paper. In order to state the theorem we provide some notation: For $x \in \mathbb{R}$ let $[x]$ be the greatest-integer function $\max\{n \in \mathbb{Z};\ n \leq x\}$ and $B(x)$ the function $x - [x]$. Denote by $\sim_\Gamma$ the following equivalence relation on $\mathcal{M}(n)$:

$$M \sim_\Gamma M' \iff \begin{cases} M \text{ and } M' \text{ are } \Gamma\text{-conjugate,} \\ M \text{ and } M' \text{ are parabolic and } \gamma M \text{ is } \Gamma\text{-conjugate} \\ \text{to } M' \text{ for some } \gamma \in \Gamma_M, \end{cases}$$

where $\Gamma_M = \{\gamma \in \Gamma;\ \gamma M = M\gamma\}$ is the centraliser of M. We also define $\delta_1(n) := 1$ if $n = m^2$ for some integer m and $\delta_1(n) := 0$ otherwise and write m^{-1} for the inverse of m in $(\mathbb{Z}/N\mathbb{Z})^*$.

THEOREM 6.3. *Let L be an even non-degenerate lattice with* $\operatorname{sig}(L)$ *even, $k \in \mathbb{Z}$ with $k > 2$ and $n \in \mathbb{N}$ be as in* (4.16). *Furthermore, let $\mathcal{M}(n)$ and Γ be defined as in* (4.18) *and* (4.17)*, respectively. Then the trace of the Hecke operator $T(n)$ on the space $S_{k,L}$ is given by*

$$\operatorname{tr}(T(n)) = A_1 + A_2 + A_3,$$

where

$$A_1 = \delta_1(n) n^{-1} \frac{k-1}{24} \frac{g_m(L)}{g(L)} \left(|L'/L| + e(-\operatorname{sig}(L)/4)(-1)^k |(L'/L)_2| \right) \tag{6.3}$$

is the contribution of the scalar matrices,

$$A_2 = -\frac{\delta_1(n)}{4} n^{k/2-1} \frac{g_m(L)}{g(L)} \sum_{v \in \mathbb{Z}/m\mathbb{Z}} (C_1(v) + C_2(v)) \tag{6.4}$$

with

$$C_1(v) = \sum_{\mu \in L'/L} e(-vm^{-1}\mu^2/2) e\Big(-\frac{v}{m} B(-\mu^2/2) \Big) \times \begin{cases} (1 - 2B(-\mu^2/2)), & \textit{if } \frac{v}{m} \in \mathbb{Z}, \\ \frac{2}{1 - e(-\frac{v}{m})}, & \textit{if } \frac{v}{m} \notin \mathbb{Z} \end{cases} \tag{6.5}$$

and

$$\begin{aligned} C_2(v) = e\Big(-\frac{\operatorname{sig}(L)}{4} \Big)(-1)^k \sum_{\mu \in (L'/L)_2} e(-vm^{-1}\mu^2/2) e\Big(-\frac{v}{m} B(-\mu^2/2) \Big) \\ \times \begin{cases} (1 - 2B(-\mu^2/2)), & \textit{if } \frac{v}{m} \in \mathbb{Z}, \\ \frac{2}{1 - e(-\frac{v}{m})}, & \textit{if } \frac{v}{m} \notin \mathbb{Z} \end{cases} \end{aligned} \tag{6.6}$$

is the contribution of the parabolic matrices and

$$A_3 = n^{k/2-1}\Bigg(\sum_{\substack{t\in\mathbb{Z}\\ D:=t^2-4n<0}} \sum_{\substack{a,b,c\in\mathbb{Z}\\ |b|\le a\le c\\ b^2-4ac=D}} 2\,\mathrm{Re}\left(C_k^{-1} I\big(A_{[a,b,c]}\big)\,\mathrm{tr}\left(\rho_L^{-1}\big(A_{[a,b,c]}\big)\right)\right)$$

$$+ \sum_{\substack{t\in\mathbb{Z}\\ t^2-4n=u^2}} \sum_{0\le b<u} I\big(A_{[b,u]}\big)\Big(\mathrm{tr}\left(\rho_L^{-1}\big(A_{[b,u]}\big)\right) + \mathrm{tr}\left(\rho_L^{-1}\big(-A_{[b,u]}\big)\right)\Big)\Bigg) \qquad (6.7)$$

is the contribution of the elliptic and hyperbolic matrices. Here, $A_{[a,b,c]}$ *and* $A_{[b,u]}$ *are the matrices* $\begin{pmatrix} \frac{1}{2}(t-b) & -c \\ a & \frac{1}{2}(t+b) \end{pmatrix}$ *and* $\begin{pmatrix} \frac{1}{2}(t-u) & b \\ 0 & \frac{1}{2}(t+u) \end{pmatrix} \in \mathcal{M}(n)$*, respectively, and the numbers* $I(A_{[a,b,c]})$ *and* $I(A_{[b,u]})$ *are given by*

$$\begin{aligned} C_k^{-1} I(A_{[a,b,c]}) &= \begin{cases} \frac{1}{|\Gamma_{A_{[a,b,c]}}|}\frac{\rho^{1-k}}{\rho-\overline{\rho}}, & \text{if } c>0,\\ \frac{1}{|\Gamma_{A_{[a,b,c]}}|}\frac{\overline{\rho}^{1-k}}{\overline{\rho}-\rho}, & \text{if } c<0,\end{cases}\\ C_k^{-1} I(A_{[b,u]}) &= \frac{1}{2}\left(\frac{t-u}{t+u}-1\right)^{-1}\left(\frac{t-u}{t+u}\right)^{k/2}, \end{aligned} \qquad (6.8)$$

where $\rho = \frac{t}{2} + i\frac{\sqrt{|D|}}{2}$ *and* $\overline{\rho} = \frac{t}{2} - i\frac{\sqrt{|D|}}{2}$ *are the roots of* $x^2 - tx + n$.

REMARK 6.4.

(a) The expressions $C_1(v)$ and $C_2(v)$ are independent of the choice of the representative v which can be verified by a straightforward computation. Further, the summands of the sums over L'/L in $C_1(v)$ and $C_2(v)$ are independent of the choice of the representative μ since $B(x+l) = B(x)$ for any $l \in \mathbb{Z}$.

(b) The expressions $I(A_{[a,b,c]})$ and $I(A_{[b,u]})$ in (6.8) are up to the factor $2C_k^{-1}$ the expressions $I(\beta)$ in [**23**], Theorem 2, and therefore well defined.

Proof. The proof of the theorem is based on the proof of Theorem 1 in [**35**], pp. 183–184 and the arguments of [**23**], Chapter 2 and Chapter 3.

In the following, we compute the contribution to the trace of the elliptic, hyperbolic, parabolic and cuspidal-hyperbolic elements to the trace. In order to simplify the formulae in the proof, we use the notation

$$f_M(\tau) = \phi(M,\tau)^{-2k}(M\tau - \overline{\tau})^{-k}.$$

Since the Weil representation on $\mathcal{M}(n)$ factors through a subgroup of the *finite* group $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ the factor $\langle \rho_L^{-1}(M)\mathfrak{e}_\mu, \mathfrak{e}_\mu\rangle$ is bounded for all $M \in \mathcal{M}(n)$. Therefore, all matters of convergence do not depend on the factor coming from the Weil representation and can be treated as in the case of the scalar valued trace formula. In fact, all of the following steps can be justified with the same arguments given in [**23**] or [**39**].

The contribution of the scalar elements: It is easy to see that $\mathcal{M}(n)$ does not contain any scalar elements unless $n = m^2$. If $n = m^2$ then $\mathcal{M}(n)$ contains exactly the two scalar elements that are represented by

$$E_m = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \text{ and } -E_m = \begin{pmatrix} -m & 0 \\ 0 & -m \end{pmatrix}$$

which belong to two different conjugacy classes. Their contribution to the trace is

$$\left(\sum_{\mu \in L'/L} \langle \rho_L^{-1}(E_m)\mathfrak{e}_\mu, \mathfrak{e}_\mu \rangle + (-1)^k \sum_{\mu \in L'/L} \langle \rho_L^{-1}(-E_m)\mathfrak{e}_\mu, \mathfrak{e}_\mu \rangle \right) \int_{\Gamma(1)\backslash\mathbb{H}} (2imy)^{-k} y^k d\mu(\tau)$$
$$= \left(|L'/L| + e(-\operatorname{sig}(L)/4)(-1)^k |(L'/L)_2|\right) \frac{g_m(L)}{g(L)} (2mi)^{-k} \operatorname{vol}(\Gamma(1)\backslash\mathbb{H}).$$

The contribution of the elliptic and non-cuspidal hyperbolic elements: Using [**23**], Lemme 1, p. 13 and Lemma 6.1 the contribution of the elliptic and non-cuspidal elements to the trace is given by

$$\int_{\Gamma(1)\backslash\mathbb{H}} \sum_{\gamma \in \Gamma/\Gamma_A} f_A(\gamma\tau) \left(\sum_{\mu \in L'/L} \langle \rho_L^{-1}(A)\mathfrak{e}_\mu, \mathfrak{e}_\mu \rangle \right) \operatorname{Im}(\gamma\tau)^k d\mu(\tau)$$
$$= \sum_{\mu \in L'/L} \langle \rho_L^{-1}(A)\mathfrak{e}_\mu, \mathfrak{e}_\mu \rangle \int_{\Gamma(1)\backslash\mathbb{H}} \sum_{\gamma \in \Gamma/\Gamma_A} f_A(\gamma\tau) \operatorname{Im}(\gamma\tau)^k d\mu(\tau). \tag{6.9}$$

The integral appearing in this expression has already been treated by [**23**], proof of Theorem 2, pp. 17–18 and thus its value is equal to (6.8) if the matrix A is elliptic, and 0 for those hyperbolic matrices A for which $\operatorname{tr}(A)^2 - 4\det(A)$ is not a square. Furthermore, a set of representatives of the Γ-conjugacy classes of elliptic matrices in $\mathcal{M}(n)$ is given by

$$\bigcup_{\substack{t \in \mathbb{Z} \\ D = t^2 - 4n < 0}} \left\{ A_{[a,b,c]}, A_{[-a,-b,-c]}; \quad a, b, c \in \mathbb{Z} \text{ with } |b| \le a \le c \text{ and } b^2 - 4ac = D \right\}.$$

Here $[a, b, c]$ denotes the binary quadratic form $ax^2 + bxy + cy^2$. All details, in particular the relation between conjugacy classes of elliptic matrices and equivalence classes of positive and negative definite binary quadratic forms, can be found in [**23**], pp. 24–28, and [**39**], p. 49. Note that $A_{[-a,-b,-c]} = \det(A_{[a,b,c]})A_{[a,b,c]}^{-1}$. Therefore, by (4.2) we obtain

$$\rho_L^{-1}(A_{[-a,-b,-c]}) = \frac{g_1(L)}{g_n(L)} \rho_L(A_{[a,b,c]}) = \rho_L(A_{[a,b,c]}). \tag{6.10}$$

The last equation holds since $g_n(L) = g_{t^2}(L) = g_1(L)$, which can be verified using definition (2.5) of the sum $g_n(L)$ and the fact that n is coprime to the level N. Equation (6.10) yields

$$C_k^{-1} \operatorname{tr}(\rho_L^{-1}(A_{[a,b,c]}))I(A_{[a,b,c]}) + C_k^{-1} \operatorname{tr}(\rho_L^{-1}(A_{[-a,-b,-c]}))I(A_{[-a,-b,-c]})$$
$$= C_k^{-1} \operatorname{tr}(\rho_L^{-1}(A_{[a,b,c]}))I(A_{[a,b,c]}) + \overline{C_k^{-1} \operatorname{tr}(\rho_L^{-1}(A_{[a,b,c]}))I(A_{[a,b,c]})}.$$

The contribution of the hyperbolic elements stabilising cusps: An element A of $\mathcal{M}(n)$ has cusps as fixed points if and only if $\mathrm{tr}(A) - 4n$ is a square in $\mathbb{Z}$. Moreover, a complete set of representatives of Γ-conjugacy classes of hyperbolic elements of $\mathcal{M}(n)$ is given by

$$\left\{A_{[b,u]}, -A_{[b,u]}; \quad t \in \mathbb{Z} \text{ with } t^2 - u^2 = 4n \text{ and } 0 \leq b < u\right\}$$

(see [**23**], p. 28). If A is a hyperbolic element of $\mathcal{M}(n)$ that stabilises cusps of $\mathrm{SL}_2(\mathbb{Z})$ we follow the classical approach of the trace formula to avoid convergence problems, replace the fundamental domain $\mathcal{F}$ of $\Gamma(1)$ by $\mathcal{F}(c) = \{\tau \in \mathcal{F}; \quad \mathrm{Im}(\tau) < c\}$ and consider

$$\begin{aligned} I^*(A, c) &:= \int_{\mathcal{F}(c)} \sum_{\gamma \in \Gamma_A \backslash \Gamma} \sum_{\mu \in L'/L} \langle \rho_L^{-1}(A)\mathfrak{e}_\mu, \mathfrak{e}_\mu \rangle f_A(\gamma\tau)\, \mathrm{Im}(\gamma\tau)^k d\mu(\tau) \\ &= \mathrm{tr}(\rho_L^{-1}(A)) \int_{\bigcup_{\gamma \in \Gamma_A \backslash \Gamma} \gamma \mathcal{F}(c)} f_A(\tau)\, \mathrm{Im}(\tau)^k d\mu(\tau). \end{aligned}$$

By [**23**], Lemme 1, p. 13, [**23**], p. 19, [**23**], (5), p. 12 or similar calculations to the calculations of [**40**], p. 173, respectively, we get

$$\begin{aligned} \lim_{c \to \infty} I^*(A, c) &= 2^{k-1} C_k n^{k/2}\, \mathrm{tr}(\rho_L^{-1}(A))(t+u)^{-k} \left(\frac{t-u}{t+u} - 1\right)^{-1} \\ &= \frac{C_k}{2}\, \mathrm{tr}(\rho_L^{-1}(A)) \left(\frac{t-u}{t+u}\right)^{k/2} \left(\frac{t-u}{t+u} - 1\right)^{-1} \end{aligned}$$

if $A = \begin{pmatrix} \frac{t-u}{2} & b \\ 0 & \frac{t+u}{2} \end{pmatrix}$.

The contribution of the parabolic matrices: We assume that $n = m^2$ since otherwise $\mathcal{M}(n)$ does not contain any parabolic elements. Then a set of conjugacy classes can be given by

$$\begin{pmatrix} m - v & v \\ -v & m + v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m & v \\ 0 & m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \tag{6.11}$$

with $v \in \mathbb{Z}/m\mathbb{Z}$, $m > 0$ (cf. [**23**], pp. 29–30 with $c = x = X = 1$). Moreover, the centraliser of $A_v = \left(\begin{smallmatrix} m & v \\ 0 & m \end{smallmatrix}\right)$ is

$$\Gamma_{A_v} = \left\{\pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}; \quad t \in \mathbb{Z}\right\}. \tag{6.12}$$

We obtain that the contribution of the parabolic elements to the trace can be written as

$$\sum_{v \in \mathbb{Z}/m\mathbb{Z}} \int_{\Gamma(1)\backslash\mathbb{H}} \sum_{\gamma \in \Gamma_{A_v} \backslash \Gamma} \sum_{\gamma' \in \Gamma'_{A_v}} f_{\gamma^{-1}\gamma' A_v \gamma}(\tau) \left(\sum_{\mu \in L'/L} \langle \rho_L^{-1}(\gamma^{-1}\gamma' A_v \gamma)\mathfrak{e}_\mu, \mathfrak{e}_\mu \rangle \right) \mathrm{Im}(\tau)^k d\mu(\tau)$$

with $\Gamma'_A = \left\{\gamma' \in \Gamma_A; \quad A\gamma' \neq \alpha E_1,\ \alpha \neq 0\right\}$. Because of convergence problems, we work again with a cut-off fundamental domain and in order to determine the contribution of the parabolic elements to the trace we then have to determine the limit of

$$I(P,c) := \sum_{v\in\mathbb{Z}/m\mathbb{Z}} \int_{\mathcal{F}_\infty(c)} \sum_{\gamma'\in\Gamma'_{A_v}} f_{\gamma' A_v}(\tau)\left(\sum_{\mu\in L'/L} \langle \rho_L^{-1}(\gamma' A_v)\mathfrak{e}_\mu, \mathfrak{e}_\mu\rangle\right) \operatorname{Im}(\tau)^k d\mu(\tau) \quad (6.13)$$

as $c\to\infty$. Here $\mathcal{F}_\infty(c) := \{z\in\mathbb{H}; 0\le \operatorname{Re}(\tau)\le 1, \operatorname{Im}(\tau)\le c\}$. Oesterlé calculated the limit of (6.13) with the contribution of the Weil representation removed (see [**23**], p. 21). In the following, we adapt his proof to our situation. By [**23**], Lemme 2, p. 14 this integral converges (cf. [**23**], p. 21).

For the evaluation of (6.13), we need to consider the cases $\gamma' = +T^r$ and $\gamma' = -T^r \in \Gamma'_{A_v}$ separately. In particular, these two cases have to be considered when we compute the character of the Weil representation. With the help of the Lemmas 6.1 and 6.2

$$\sum_{\mu\in L'/L} \langle \rho_L^{-1}(+T^r A_v)\mathfrak{e}_\mu, \mathfrak{e}_\mu\rangle = \frac{g_m(L)}{g(L)} \sum_{\mu\in L'/L} e(-r\mu^2/2)e(-vm^{-1}\mu^2/2).$$

Similarly,

$$\sum_{\mu\in L'/L} \langle \rho_L^{-1}(-T^r A_v\mathfrak{e}_\mu, \mathfrak{e}_\mu\rangle = \frac{g_m(L)}{g(L)} e(-\operatorname{sig}(L)/4) \sum_{\mu\in(L'/L)_2} e(-r\mu^2/2)e(-vm^{-1}\mu^2/2).$$

[**23**], Lemme 1, p. 13 and [**11**], Lemma 5.5.1 (a) yield for $I(P,c)$ in the “+ case”

$$\begin{aligned}
\lim_{c\to\infty} I(P,c,+) &= \frac{g_m(L)}{g(L)} \sum_{v\in\mathbb{Z}/m\mathbb{Z}} \sum_{\mu\in L'/L} e(-vm^{-1}\mu^2/2) \\
&\quad\times \lim_{c\to\infty} \int_{\mathcal{F}_\infty(c)} \sum_{r\in\mathbb{Z}'} e(-r\mu^2/2) f_{T^r A_v}(\tau)\operatorname{Im}(\tau)^k d\mu(\tau) \\
&= \frac{g_m(L)}{g(L)} \sum_{v\in\mathbb{Z}/m\mathbb{Z}} \sum_{\mu\in L'/L} e(-vm^{-1}\mu^2/2) \\
&\quad\times \lim_{c\to\infty} \int_{\mathcal{F}_\infty(c)} \sum_{r\in\mathbb{Z}'} e(rB(-\mu^2/2))\Big(2i\operatorname{Im}(\tau) + \frac{v}{m} + r\Big)^{-k} \operatorname{Im}(\tau)^k d\mu(\tau)
\end{aligned} \quad (6.14)$$

where $\mathbb{Z}' = \{r\in\mathbb{Z};\ r\ne -\frac{v}{m}\}$. The last line of (6.14) can be evaluated in exactly the same way as in [**23**], p. 21 and p. 43 and we obtain:

$$\begin{aligned}
\lim_{c\to\infty} I(P,c,+) &= \frac{2^{1-k} i^{2-k}\pi}{k-1} \frac{g_m(L)}{g(L)} \sum_{v\in\mathbb{Z}/m\mathbb{Z}} \sum_{\mu\in L'/L} e(-vm^{-1}\mu^2/2) e\Big(-\frac{v}{m} B(-\mu^2/2)\Big) \\
&\quad\times \begin{cases} (1-2B(-\mu^2/2)), & \text{if } \frac{v}{m}\in\mathbb{Z}, \\ \frac{2}{1-e(-\frac{v}{m})}, & \text{if } \frac{v}{m}\notin\mathbb{Z}. \end{cases}
\end{aligned}$$

The same argumentation as above yields in the “– case” formula (6.6). □

THEOREM 6.5. *Under the same conditions as in Theorem 6.3 the dimension of the space $S_{k,L}$ of vector-valued cusp forms of weight k and type ρ_L is given by*

$$\begin{aligned}
&\dim_{\mathbb{C}}(S_{k,L}) \\
&= -\frac{1}{4}\left(\sum_{\mu\in L'/L}(1-2B(-\mu^2/2)) + e(-\operatorname{sig}(L)/4)(-1)^k \sum_{\mu\in(L'/L)_2}(1-2B(-\mu^2/2))\right) \\
&\quad + \frac{k-1}{24}\left(|L'/L| + e(-\operatorname{sig}(L)/4)(-1)^k|(L'/L)_2|\right) + 2\operatorname{Re}\left(\frac{(-i)^k}{8}\operatorname{tr}(\rho_L^{-1}(S))\right) \\
&\quad + 2\operatorname{Re}\left(\frac{e\left(\frac{1-k}{6}\right)}{6i\sqrt{3}}\operatorname{tr}\left(\rho_L^{-1}(ST)\right)\right) + 2\operatorname{Re}\left(\frac{e\left(\frac{1-k}{3}\right)}{6i\sqrt{3}}\operatorname{tr}\left(\rho_L^{-1}(T^{-1}S)\right)\right).
\end{aligned} \tag{6.15}$$

Proof. It is evident from the definition of the Hecke operator on the space of vector-valued cusp forms that $\operatorname{tr}(T(1)) = \dim_{\mathbb{C}}(S_{k,L})$. Thus it is easy to see that the first three terms in (6.15) are the terms A_1 and A_2 of the trace formula in Theorem 6.3. In order to treat the contribution from the elliptic matrices, we note that the only elliptic matrices in Γ are those with trace $t = -1, 0, 1$. The correspondence between Γ-conjugacy classes of elliptic matrices in Γ with trace t and equivalence classes of positive definite binary quadratic forms of discriminant $D = t^2 - 4$ (see e.g. [**23**], p. 24–25, or [**39**], p. 49, respectively) yields that the following representatives contribute to the dimension formula:

$$t = 0 : \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right),\ t = 1 : \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right),\ t = -1 : \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right).$$

Hence the last three expressions of (6.15) are a direct consequence of (6.7) and (6.8). Furthermore, one can easily prove that the equation $t^2 - 4 = u^2$ has for $u \neq 0$ no solution $(u, t) \in \mathbb{Z}^2$ using the formula for generating Pythagorean triples. Thus, the theorem follows. □

ACKNOWLEDGEMENTS. We would like to thank J. Bruinier and S. Ehlen for helpful comments. Part of this research was done while N. Raulf was a guest at the Max-Planck-Institut für Mathematik in Bonn and she wants to thank the Max-Planck-Institut for its hospitality. We also want to thank the referee for many helpful comments.

REFERENCES

1. P. Bauer-Price, *The Selberg trace formula for* PSL(2, $\mathcal{O}_K$) *for imaginary quadratic number fields K of arbitrary class number,* Bonner Mathematische Schriften, vol. 221 (Universität Bonn, Mathematisches Institut, Bonn, 1991).

2. B. Berndt, R. Evans and K. Williams, *Gauss and Jacobi sums*, Canadian Mathematical Society series of monographs and advanced texts, vol. 21 (Wiley, New York, NY, 1998).

3. R. Borcherds, Automorphic forms with singularities on Grassmannians, *Inv. Math.* **132** (1998), 491–562.

4. R. Borcherds, Reflection groups of Lorentzian lattices, *Duke Math. J.* **104** (2000), 319–366.

5. K. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87 (Springer-Verlag, Berlin, 1982).

6. J. H. Bruinier, *Borcherds products on* $O(2, l)$ *and Chern classes of Heegner divisors*, Springer Lecture Notes in Mathematics vol. 1780 (Springer-Verlag, Berlin, 2002).

7. J. H. Bruinier, On the rank of Picard groups of modular varieties attached to orthogonal groups, *Compos. Math.* **133** (2002), 49–63.

8. J. H. Bruinier and M. Bundschuh, On Borcherds products associated with lattices of prime discriminant, *Ramanujan J.* **7** (2003), 49–61.

9. J. H. Bruinier and K. Ono, Algebraic formulas for the coefficients of half-integral weight harmonic weak Maass forms, preprint (2011).

10. J. H. Bruinier and O. Stein, The Weil representation and Hecke operators on vector valued modular forms, *Math. Z.* **264**, 249–270.

11. F. Diamond and J. Shurman, *A first course in modular forms*, Graduate Texts in Mathematics vol. 228 (Springer-Verlag, New York, 2005)

12. W. Ebeling, *Lattices and codes*, A course partially based on lectures by F. Hirzebruch, Second revised edition, Advanced Lectures in Mathematics (Vieweg, Braunschweig, 2002).

13. I. Y. Efrat, The Selberg trace formula for $PSL_2(\mathbb{R}^n)$, *Mem. Amer. Math. Soc.* **65**(359) (1987).

14. W. Eholzer and N.-P. Skoruppa, Modular invariance and uniqueness of conformal characters, *Commun. Math. Phys.* **174** (1995), 117–136.

15. J. Elstrodt, F. Grunewald and J. Mennicke, *Groups acting on hyperbolic space*, Harmonic analysis and number theory, Springer Monographs in Mathematics (Springer-Verlag, Berlin, 1998).

16. H. Hagemeier, *Automorphe Produkte singulären Gewichts*, Dissertation (TU Darmstadt 2010).

17. D. Hejhal, *The Selberg trace formula for* $PSL(2, \mathbb{R})$, volume 1, Lecture Notes in Mathematics, **548** (Springer Verlag, Berlin 1976).

18. D. Hejhal, *The Selberg trace formula for* $PSL(2, \mathbb{R})$, volume 2, Lecture Notes in Mathematics **1001** (Springer Verlag, Berlin, 1983).

19. H. Hijikata, Explicit formula of the traces of Hecke operators for $\Gamma_0(N)$, *J. Math. Soc. Japan* **26** (1974), 56–82.

20. X.-J. Li, *On the trace of Hecke operators for Maass forms*, Number theory (Ottawa, ON, 1996), 215–229, CRM Proc. Lecture Notes, vol. 19 (Amer. Math. Soc., Providence, RI, 1999).

21. J. Milnor and D. Husemöller, *Symmetric bilinear forms*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 73 (Springer-Verlag, New York-Heidelberg, 1973).

22. V. L. Kalinin, A trace formula for Hecke operators, *Math. USSR Sbornik,* **47**(2) (1984).

23. J. Oesterlé, *Sur la trace des operateurs de Hecke*, Thèse de 3e cycle (Université de Paris-Sud, University in Orsay, France, 1977).

24. N. Raulf, Traces of Hecke operators acting on three-dimensional hyperbolic space, *J. Reine Angew. Math.* **591** (2006), 111–148.

25. P. Sarnak, Class numbers of indefinite binary quadratic forms, *J. Number Theory* **15** (1982), 229–247.

26. P. Sarnak, The arithmetic and geometry of some hyperbolic three-manifolds, *Acta Math.* **151**(3-4), (1983), 253–295.

27. P. Sarnak, *Statistical properties of eigenvalues of the Hecke operators,* Analytic number theory and Diophantine problems (Stillwater, OK, 1984), 321–331, Progr. Math., vol. 70 (Birkhäuser Boston, Boston, MA, 1987).

28. N. R. Scheithauer, The Weil representation of $SL_2(\mathbb{Z})$ and some applications, *Int. Math. Res. Not.* **8** (2009), 1488–1545.

29. N. R. Scheithauer, *Moonshine for Conway's group* (Habilitationsschrift, Ruprecht-Karls-Universität Heidelberg, 2004).

30. N. R. Scheithauer, On the classification of automorphic products and generalized Kac-Moody algebras, *Invent. Math.* **164**, 641–678 (2006).

31. A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, *J. Indian Math. Soc. B* **20** (1956), 47–87.

32. J.-P. Serre, Répartition asymptotique des valeurs propres de l'opérateur de Hecke T_p, *J. Amer. Math. Soc.* **10**(1) (1997), 75–102.

33. H. Shimizu, On traces of Hecke operators, *J. Fac. Sci. Univ. Tokyo* **10** (1963), 1–19.

34. T. Shintani, On the construction of holomorphic cusp forms of half integral weight, *Nagoya Math. J.* **58** (1975), 83–126.

35. N.-P. Skoruppa and D. Zagier, A trace formula for Jacobi forms, *J. Reine Angew. Math.* **393** (1989), 168–198.

36. F. Strömberg, *Weil representations associated to finite quadratic modules*, *Math. Z.* **275** (2013), 509–527.

37. T. Teruji, The character of the Weil representation, *J. Lond. Math. Soc., II. Ser.* **77**(1) (2008), 221–239.

38. F. Werner, Discriminant forms and Hecke operators, Master Thesis (TU Darmstadt, 2010).

39. D. Zagier, *The Eichler-Selberg trace formula on* $\mathrm{SL}_2(\mathbb{Z})$, Appendix to S. Lang, Introduction to Modular Forms, Grundlehren d. math. Wiss., vol. 222 (Springer-Verlag, Berlin-Heidelberg-New York, 1976), 44–54.

40. D. Zagier, Correction to "the Eichler-Selberg trace fomula on $\mathrm{SL}_2(\mathbb{Z})$", in *Modular functions of one variable VI* Lecture Notes in Mathematics, vol. 627 (Springer-Verlag, Berlin, 1977), 171–173.