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ANALYTIC PROPERTIES OF EISENSTEIN SERIES AND STANDARD L-FUNCTIONS

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Abstract. We prove a functional equation for a vector valued real analytic Eisenstein series transforming with the Weil representation of $\mathrm{Sp}(n,\mathbb{Z})$ on $\mathbb{C}[(L'/L)^n]$. By relating such an Eisenstein series with a real analytic Jacobi Eisenstein series of degree n, a functional equation for such an Eisenstein series is proved. Employing a doubling method for Jacobi forms of higher degree established by Arakawa, we transfer the aforementioned functional equation to a zeta function defined by the eigenvalues of a Jacobi eigenform. Finally, we obtain the analytic continuation and a functional equation of the standard L-function attached to a Jacobi eigenform, which was already proved by Murase, however in a different way.

§1. Introduction

Let L be an even lattice of rank $m \in 2\mathbb{Z}$ and of signature (b^+, b^-), equipped with a bilinear form $(\cdot,\cdot)$ and associated quadratic form q. By L' we denote the dual lattice of L. For $n \in \mathbb{N}$, let $\mathrm{Sp}(n,\mathbb{Z})$ be the symplectic group of degree n over $\mathbb{Z}$, and let $\mathbb{H}_n$ be the Siegel upper half space. There is a unitary representation $\rho_{L,n}$ of $\mathrm{Sp}(n,\mathbb{Z})$ on the group ring $\mathbb{C}[(L'/L)^n]$; see (3.30) and the remarks afterward. Depending on $\tau \in \mathbb{H}_n$ and $s \in \mathbb{C}$, the non-holomorphic Siegel Eisenstein series of genus n associated to the Weil representation $\rho_{L,n}$ is given by

$$E^n_{l,0}(\tau,s) = \det(\mathrm{Im}(\tau))^s \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{Sp}(n,\mathbb{Z})} |\det(J(\gamma,\tau))|^{-2s} \det(J(\gamma,\tau))^{-l} \rho^{-1}_{L,n}(\gamma)\mathfrak{e}_0,$$

where $l \in \mathbb{N}$ is assumed to be even, $\mathfrak{e}_0 \in \mathbb{C}[(L'/L)^n]$, $J(\gamma,\tau) = c\tau + d$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$, and $\Gamma_\infty = \left\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{Sp}(n,\mathbb{Z}) \mid c = 0\right\}$. For $n = 1$ and a certain lattice L, the analytic properties of this series have been studied in [BY, Chapter 2]. It is also a generalization of the vector valued Eisenstein series discussed in [BK, Chapter 3], and the classical non-holomorphic Siegel Eisenstein series. Eisenstein series play an important role in number theory. For example, Eisenstein series are one of the most important tools in the study of L-functions. In particular, by means of the doubling method [Bo, Ga], the analytic continuation and functional equations of L-functions can be proved. The doubling method exploits the fact that the involved Eisenstein series satisfy the very same analytic properties. In a forthcoming paper, we will study the analytic properties of the standard L-function of a vector valued modular form transforming with the Weil representation $\rho_{L,1}$. To this end, we developed a doubling method that is tailored to the case of vector valued Siegel modular forms transforming with the Weil representation $\rho_{L,2}$; see [St]. Therefore, it is of great interest to study the analytic properties of $E^n_{l,0}$.

Roughly, this paper consists of two parts. The first one aims at proving the meromorphic continuation to the complex s-plane and a functional equation of $E^n_{l,0}(\tau,s)$. This may be

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considered as a generalization of Kalinin's result in [Kl] and the corresponding result in [BY]. In fact, our proof generalizes the one given in [BY, Proposition 2.5]: the idea is to relate $E^n_{l,0}(\tau, s)$ to a vector valued adelic Eisenstein series

$$E_L(\tau, s, l) = \det(\mathrm{Im}(\tau))^{-l/2} \sum_{\mu\in(L'/L)^n} E(g_0, s, \Phi^l_\mu)\varphi_\mu,$$

where each component is an adelic Eisenstein series associated to a special standard section Φ^l_μ. By Langland's theory [At], each such Eisenstein series has the desired analytic properties. In particular, it satisfies a functional equation, which involves the global intertwining operator

$$M(s)\Phi(g, s) = \int_{N(\mathbb{A})} \Phi(wng, s)\, dn.$$

Here we have $g \in \mathrm{Sp}(n, \mathbb{A})$, and $N(\mathbb{A})$ and w are defined in Section 3.1. The integral converges absolutely for $\mathrm{Re}(s) > (n + 1)/2$. In order to obtain a more explicit form of the functional equation, we calculate for each prime p the local integral $\int_{N(\mathbb{Q}_p)} \Phi(wng, s)\, dn$. For all unramified primes, as well as for $p = \infty$, there are explicit formulas available; see for example [KR] or Section 3.5.1. For the ramified primes, we follow an idea of Kudla [Ku2, Lemma A.3] to express $M(s)\Phi^l_\mu(g, s)$ as a local Whittaker integral

$$\int_{\mathrm{Sym}_n(\mathbb{Q}_p)} \int_{\mu+L^n_{p,r}} \psi_p\left(\frac{1}{2}\,\mathrm{tr}(bQ^r[x])\right) dx\, db,$$

which is closely related to local densities; see [Ya]. Provided L'/L is *anisotropic* and $|L'/L|$ is *odd*, we evaluate this integral explicitly (see Lemma 3.9) by means of a formula of Hironaka and Sato [HS], which gives a very general expression for the local densities for an odd prime p and arbitrary $n \in \mathbb{N}$ in terms of invariants of quadratic forms. In fact, parts of their formula can be found in a modified form in our functional equation for $E^n_{l,0}(\tau, s)$. Combining all this, Theorem 3.16 states the desired functional equation. Although the assumption that L'/L is anisotropic results in simpler formulas, our functional equation due to the local ramified factors is still quite complicated. But for special cases, we obtain clean and quite simple formulas; see for example [St].

The second part of the paper can be seen as an application of the results of the first part. It is concerned with Jacobi forms of higher degree and aims at proving a functional equation for the standard L-function $L(s, f)$ of a Jacobi eigenform f (see Section 4.3 for the definition of $L(s, f)$). Note that Murase has already established such a functional equation for $L(s, f)$ in [Mu1, Mu2], however by a completely different approach. Our functional equation involves quite complicated local factors coming from formulas of the above-mentioned local densities and Siegel series (see the remarks after Proposition 4.8 for details). These local factors may cancel each other out. But this seems not obvious and it might therefore not be easy to prove that both functional equations coincide.

To employ our results of the first part, the observation that there is an isomorphism between the space $J_{l,S}(\Gamma_{n,m})$ of Jacobi forms of degree n, index $S \in \mathrm{Sym}_m(\mathbb{Q})$, and weight l, as studied in [Zi, Mu1, Mu2, Ar2], and the space of vector valued Siegel modular forms for the Weil representation $\rho_{L,n}$ is crucial. Based on this isomorphism, it is possible to express the real analytic Jacobi Eisenstein series

$$E^{(n)}_{l,S}(\tau, z; s) = \det(\mathrm{Im}(\tau))^s \sum_{\gamma\in\Gamma^\infty_{n,m}\backslash\Gamma_{n,m}} J_{S,l}(\gamma, (\tau, z))^{-1} |\det(J(M, \tau))|^{-2s},$$

$$(\tau, z) \in \mathbb{H}_n \times M_{m,n}(\mathbb{C}),$$

in terms of the vector valued Eisenstein series $E^n_{l,0}$. Here M is the $\mathrm{Sp}(n)$-component of γ, $J_{S,l}(\gamma,(\tau,z))$ is the standard automorphy factor, and $\Gamma^\infty_{n,m}$ is a subgroup of the Jacobi group $\Gamma_{n,m}$. See Section 4.1 for the details. The relation between both series allows us to transfer the analytic properties of $E^n_{l,0}$, mentioned above, to $E^{(n)}_{l,S}(\tau,z;s)$, as is stated in Theorem 4.2. To the best of my knowledge, a functional equation for $E^{(n)}_{l,S}(\tau,z;s)$ is so far not available in the literature and may be of independent interest.

Let D be an elementary divisor matrix, and let $f\in J^{\mathrm{cusp}}_{l,S}(\Gamma_{n,m})$ be a common eigenform of the Hecke operators $T\begin{pmatrix} D & 0\\ 0 & (D^{-1})^t\end{pmatrix}$, that is,

$$T\begin{pmatrix} D & 0\\ 0 & (D^{-1})^t\end{pmatrix} f=\lambda(f,D)f$$

for all matrices D. Then we can attach to f the Dirichlet series

$$Z_n(s,f)=\sum_D \lambda(f,D)\det(D)^{-s},$$

which is absolutely convergent for $\mathrm{Re}(s)>2n+m+1$. Arakawa proved in [Ar2] a pullback formula for the Jacobi Eisenstein series and an integral representation for $Z_n(s,f)$, as Böcherer did in [Bo] for Siegel modular forms. It is then a simple task to deduce the meromorphic continuation to the whole s-plane and a functional equation for $Z_n(s,f)$ along the same lines as in [Bo], which has—as far as I know—not appeared before in the literature.

Finally, under a certain condition imposed on the p-adic lattice $\mathbb{Z}_p^m$, with respect to the quadratic form $q(x)=x^tSx$ for all primes p, Bouganis and Marzec provide an equation that connects $Z_n(s,f)$ with the standard L-series $L(s,f)$ [BM, Chapter 7]. This equation holds in far greater generality than that needed in the present paper and simplifies to the corresponding equations (4.29) and (4.30) in our setting. It allows us to prove the desired analytic properties of $L(s,f)$, which can be found in Theorem 4.10. Note that Bouganis and Marzec proved the meromorphic continuation of $L(s,f)$ in their more general setting. It seems conceivable that one could prove a functional equation for the more general L-function along the lines of the present paper. This will be the subject of forthcoming work.

§2. Notation

We use the symbol $e(x)$, $x\in\mathbb{C}$, as abbreviation for $\exp(2\pi i x)$. By $|M|$ we mean the order of the set M. For $x\in\mathbb{R}$, $[x]$ means the integer satisfying $0\leqslant x-[x]\leqslant 1$ and for a rational prime p, we denote by $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$) the ring of p-adic integers (resp. the field of p-adic numbers); $|\cdot|_p$ is the p-adic absolute value and $\mathrm{ord}_p(\cdot)$ is the p-adic valuation of $\mathbb{Q}_p$. We write $\mathbb{A}$ (resp. $\mathbb{A}^\times$) for the adele ring (resp. the idele group) of $\mathbb{Q}$ and $\mathbb{A}_f$ for the set of finite adeles. In this paper, by m and n we always mean natural numbers. For any ring R, $M_{m,n}(R)$, R^n, and $\mathrm{Sym}_n(R)$ are the set of $m\times n$ matrices, the set of row vectors of size n, and the set of symmetric matrices in $M_{n,n}(R)$, respectively. For $a_1,a_2,\ldots,a_n\in R$, denote by $\mathrm{diag}(a_1,a_2,\ldots,a_n)$ the diagonal matrix with diagonal entries $a_1,a_2,\ldots,a_n$. We write 1_n for the unit matrix of size n and A^t for the transposed matrix of A. Also, for any matrix $S\in\mathrm{Sym}_n(\mathbb{R})$, by $S>0$ (resp. $S\geqslant 0$) we mean that S is positive (resp. semipositive) definite. As usual, $\mathrm{Sp}(n)$ is the symplectic group of degree n. $\mathrm{Sp}(n,\mathbb{R})$ acts on the upper half space $\mathbb{H}_n=\{\tau\in\mathrm{Sym}_n(\mathbb{C})\mid \mathrm{Im}(\tau)>0\}$. For $S\in\mathrm{Sym}_m(R)$ and $u,v\in M_{m,n}(R)$, we introduce the following notation:

$$S(u,v) = u^t S v \qquad \text{and} \qquad S[v] = v^t S v. \tag{2.1}$$

In accordance with [Sh], we use the notation

$$\Gamma_n(s) = \pi^{n(n-1)/4} \prod_{k=0}^{n-1} \Gamma\left(s - \frac{k}{2}\right). \tag{2.2}$$

Also, for a prime p and a character χ_p of $\mathbb{Q}_p^\times$, we define local zeta functions and L-functions, respectively:

$$\zeta_p(s) = (1 - p^{-s})^{-1},$$
$$L_p(s, \chi_p) = (1 - \chi_p(p) p^{-s})^{-1}.$$

Here $L_p(s, \chi_p)$ is defined to be equal to 1 if χ_p is ramified. Finally, by $\left(\frac{\cdot}{p}\right)$ we mean the Legendre symbol.

§3. Analytic properties of Eisenstein series

In this section, the functional equation of some vector valued non-holomorphic Eisenstein series transforming with the Weil representation is proved.

The following three subsections have the purpose of introducing the necessary theory about the degenerate principal series and adelic Eisenstein series along with some lemmas, which will be used later on in the paper. Except these lemmas, the material is well known and can be found in different places. We mainly follow [Ku1].

3.1 Symplectic groups

We set up some further notation, which will be used this way throughout Section 3, unless it is stated otherwise. Let $(L, (\cdot,\cdot))$ be an even lattice, that is a free $\mathbb{Z}$-module of finite rank m, equipped with a symmetric bilinear form $(\cdot,\cdot)$ such that $(x,x) \in 2\mathbb{Z}$ for all $x \in L$. We assume that m is *even*, L is non-degenerate and denote its signature by (b^+, b^-). We write $V = L \otimes \mathbb{Q}$ and define by

$$L' = \{x \in V \mid (x,y) \in \mathbb{Z} \text{ for all } y \in L\}$$

the dual lattice of L. Associated to $(\cdot,\cdot)$ there is a quadratic form

$$q(x) = \tfrac{1}{2}(x,x).$$

Since L is even, we have $L \subset L'$. It can be proved that L'/L is a finite abelian group. The modulo 1 reduction of $(\cdot,\cdot)$ yields a $\mathbb{Q}/\mathbb{Z}$-valued bilinear form $(\cdot,\cdot)_L$ on L'/L. The associated quadratic form q_L is the modulo 1 reduction of q with values in $\mathbb{Q}/\mathbb{Z}$. $(L'/L, (\cdot,\cdot)_L)$ is called a *finite quadratic module* or a *discriminant form*.

To the vector space $(V, (\cdot,\cdot))$ we associate a quadratic character

$$\chi_V : \mathbb{A}^\times/\mathbb{Q}^\times \longrightarrow \mathbb{C}, \qquad x \mapsto \chi_V(x) = (x, (-1)^{m/2}\det(V))_{\mathbb{A}}, \tag{3.1}$$

where $(\cdot,\cdot)_{\mathbb{A}}$ is the global Hilbert symbol and $\det(V)$ is the Gram determinant of V. Since $V = L \otimes \mathbb{Q}$, we have $|\det(V)| = |L'/L|$. We denote by $H = \mathrm{O}(V)$ the orthogonal group over $\mathbb{Q}$ attached to $(V, (\cdot,\cdot))$. Moreover, let W be the standard vector space over $\mathbb{Q}$ of dimension

$2n$ equipped with the non-degenerate skew-symmetric bilinear form $<\cdot,\cdot>$ defined by

$$w = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

Let $G = \mathrm{Sp}(W) \cong \mathrm{Sp}(n,\mathbb{Q})$ be the symplectic group over $\mathbb{Q}$ attached to $(W, <\cdot,\cdot>)$.

More generally, we consider the symplectic group $G(\mathbb{A}) = \mathrm{Sp}(n,\mathbb{A})$. Within $G(\mathbb{A})$, we have the following subgroups:

$$M(\mathbb{A}) = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & (a^{-1})^t \end{pmatrix} \,\middle|\, a \in \mathrm{GL}_n(\mathbb{A}) \right\}, \tag{3.2}$$

$$N(\mathbb{A}) = \left\{ n(b) = \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix} \,\middle|\, b \in \mathrm{Sym}_n(\mathbb{A}) \right\}. \tag{3.3}$$

These define the Siegel parabolic subgroup $P(\mathbb{A}) = N(\mathbb{A})M(\mathbb{A})$ of $G(\mathbb{A})$, which is part of the Iwasawa decomposition

$$G(\mathbb{A}) = N(\mathbb{A})M(\mathbb{A})K \tag{3.4}$$

of $G(\mathbb{A})$, where $K = \prod_v K_v$ is the maximal compact subgroup of $G(\mathbb{A})$. Here for a non-archimedean place $v = p$, the group K_p is given by $\mathrm{Sp}(n,\mathbb{Z}_p)$. For the archimedean place $v = \infty$, we have

$$K_\infty = \left\{ k = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R}) \,\middle|\, \mathbf{k} = a + ib \in U(n) \right\}, \tag{3.5}$$

where $U(n)$ means the unitary group.

The following lemma describes the structure of K_p and might be known. For $n = 1$, this result can be found in [BY, p. 644]. However, I could not find any reference for a general $n \in \mathbb{N}$. Therefore, we give a proof here.

LEMMA 3.1. *For each prime number p, the group $\mathrm{Sp}(n,\mathbb{Z}_p)$ is generated by the set G_n of matrices*

$$\begin{array}{lll} w, & & \\ m(a), & a \in \mathrm{GL}(n,\mathbb{Z}_p), & \textit{and} \\ n(b), & b \in \mathrm{Sym}_n(\mathbb{Z}_p). & \end{array} \tag{3.6}$$

Proof. The proof is an adaptation of the one which proves [Sc, Theorem 7.5, Chapter 7]. This theorem states the same result for the group $\mathrm{Sp}(n,F)$, where F is a field. The proof proceeds by induction on n and starts with a matrix $M_n = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n,\mathbb{Z}_p)$, which is then multiplied with suitable matrices of set (3.6). It can be assumed that $A \neq 0$ since otherwise, we can replace M_n with $M_n w$. By [Ke, Theorem 4.3.4], there exist matrices $U, V \in \mathrm{GL}(n,\mathbb{Z}_p)$ such that $UAV = D$, where D is of the form $\left(\begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $S \in M_{k,k}(\mathbb{Z}_p)$ is diagonal with nonzero entries in $\mathbb{Z}_p$. Multiplying M with $\left(\begin{smallmatrix} U & 0 \\ 0 & U^{-1} \end{smallmatrix}\right)$ from the left and $\left(\begin{smallmatrix} V & 0 \\ 0 & V^{-1} \end{smallmatrix}\right)$ from the right yields

$$\begin{pmatrix} A^1 & B^1 \\ C^1 & D^1 \end{pmatrix} = \left(\begin{array}{cc|cc} S & 0 & B_1 & B_2 \\ 0 & 0 & B_3 & B_4 \\ \hline C_1 & C_2 & D_1 & D_2 \\ C_3 & C_4 & D_4 & D_4 \end{array}\right), \tag{3.7}$$

where S, B_1, C_1, D_1 are $k \times k$-matrices as in [Sc]. Since matrix (3.7) is an element of $\mathrm{Sp}(n, \mathbb{Z}_p)$, it follows that $(A^1)^t C^1 \in \mathrm{Sym}_n(\mathbb{Z}_p)$, which in turn implies $C_2 = 0$ and $C_1 \in \mathrm{Sym}_k(\mathbb{Z}_p)$. One easily checks that multiplication of (3.7) with $\left(\begin{smallmatrix} 1 & 0 \\ T & 1 \end{smallmatrix}\right)$ on the left produces

$$\begin{pmatrix} A^2 & B^2 \\ C^2 & D^2 \end{pmatrix} = \left(\begin{array}{cc|cc} S & 0 & B_1 & B_2 \\ 0 & 0 & B_3 & B_4 \\ \hline 0 & 0 & D_1 & D_2 \\ 0 & C_4 & D_3 & D_4 \end{array}\right). \tag{3.8}$$

Here T is the matrix $\left(\begin{smallmatrix} -C_1 & -C_3^t \\ C_3 & 0 \end{smallmatrix}\right) \in \mathrm{Sym}_n(\mathbb{Z}_p)$. Again, (3.8) is in $\mathrm{Sp}(n, \mathbb{Z}_p)$ and the relation $(A^2)^t D^2 - (C^2)^t B^2 = 1_n$ implies $SD_1 = 1_k$, $D_2 = 0, B_3 = 0$, and $-C_4^t B_4 = 1_{n-k}$, as is deduced in [Sc]. Exactly the same conclusions and transformations (out of (3.6)) as in [Sc] remain valid for the case of p-adic integers. As a consequence, we obtain the matrix

$$\left(\begin{array}{cc|cc} S & 0 & 0 & 0 \\ 0 & 0 & 0 & B_4 \\ \hline 0 & 0 & D_1 & 0 \\ 0 & C_4 & 0 & D_4 \end{array}\right). \tag{3.9}$$

By multiplying (3.9) from the right with

$$\left(\begin{array}{cc|cc} D_1 & 0 & 0 & 0 \\ 0 & 1_{n-k} & 0 & 0 \\ \hline 0 & 0 & S & 0 \\ 0 & 0 & 0 & 1_{n-k} \end{array}\right),$$

we finally get

$$\left(\begin{array}{cc|cc} 1_k & 0 & 0 & 0 \\ 0 & 0 & 0 & B_4 \\ \hline 0 & 0 & 1_k & 0 \\ 0 & C_4 & 0 & D_4 \end{array}\right), \tag{3.10}$$

which is the same matrix as in [Sc]. Since (3.10) is in $\mathrm{Sp}(n, \mathbb{Z}_p)$, it can be checked that $M_{n-k} = \left(\begin{smallmatrix} 0 & B_4 \\ C_4 & D_4 \end{smallmatrix}\right)$ is in $\mathrm{Sp}(n-k, \mathbb{Z}_p)$. By induction, multiplication of M_{n-k} from the left and right with (finitely many) matrices from G_{n-k} gives the identity matrix. Each used matrix corresponds to the multiplication of (3.10) with the same type of matrix in G_n and the result for the blocks $0, B_4, C_4, D_4$. □

3.2 The Weil representation

The following subsection gathers some well-known facts on the Weil representation of $G(\mathbb{A}) \times H(\mathbb{A})$. In particular, the Schrödinger model of the Weil representation with explicit formulas is given. We follow [Ku1, Chapter I.1], [Ge, Chapter 2.3], and [As].

Let

$$\psi = \prod_{p \leqslant \infty} \psi_p : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times, \qquad x = (x_p) \mapsto \psi(x) = e^{2\pi i (x_\infty - \sum_{p<\infty} x_p')} \tag{3.11}$$

be the *standard non-trivial additive* character, where x_p' is the usual partial fraction of x_p. Attached to this character and the dual pair (G, H) there is a Weil representation ω.

The Schrödinger model of ω defines an action of $G(\mathbb{A}) \times H(\mathbb{A})$ on the space $S(V(\mathbb{A})^n)$ of Schwartz functions on

$$V(\mathbb{A})^n = \{v = (v_1, \ldots, v_n) \mid v_1, \ldots, v_n \in V \otimes \mathbb{A}\}.$$

Note that since $\dim(V) = m$ is assumed to be even, we do not have to work with the metaplectic cover of $G(\mathbb{A})$. The action of ω on the generators of $G(\mathbb{A})$ can be described explicitly for $\varphi \in S(V(\mathbb{A})^n)$ by the following formulas

$$\begin{aligned} &\omega(h)\varphi(v) = \varphi(h^{-1}v), \quad h \in H(\mathbb{A}),\\ &\omega(m(a))\varphi(v) = \chi_V(\det a)|\det a|_{\mathbb{A}}^{m/2}\varphi(va), \quad m(a) \in M(\mathbb{A}),\\ &\omega(n(b))\varphi(v) = \psi(\operatorname{tr} bQ[v])\varphi(v), \quad n(b) \in N(\mathbb{A}),\\ &\omega(w)\varphi(v) = \gamma \int_{V(\mathbb{A})^n} \psi(tr(v, w))\varphi(w)\, dw, \end{aligned} \tag{3.12}$$

where

$$(v, w) = ((v_i, w_j))_{i,j} \in \operatorname{Sym}_n(\mathbb{A}), \qquad Q[v] = \tfrac{1}{2}(v, v) \in \operatorname{Sym}_n(\mathbb{A}) \tag{3.13}$$

is the moment matrix. Note that if the bilinear form on L is given by a symmetric matrix S, it is easily seen that

$$(v, w) = v^t S w \qquad \text{and} \qquad Q[v] = \tfrac{1}{2} v^t S v$$

for $v, w \in V(\mathbb{A})^n$. Moreover, $dw = \prod_p dw_p$, where dw_p is the Haar measure on $V(\mathbb{Q}_p)^n$ for $p < \infty$, which is self-dual with respect to ψ_p. By γ we mean

$$\gamma = \prod_{p \leqslant \infty} \gamma_p. \tag{3.14}$$

Here

$$\gamma_p = \gamma_p(V)^n, \quad p < \infty, \tag{3.15}$$

$$\gamma_\infty = \gamma_\infty(V)^n, \quad p = \infty, \tag{3.16}$$

where $\gamma_p(V)$ is the local Weil index and $\gamma_\infty(V) = e((b^+ - b^-)/8)$. Note that by the product formula we have

$$\gamma_\infty(V) \prod_{p < \infty} \gamma_p(V) = 1. \tag{3.17}$$

3.3 The degenerate principal series

Using the Iwasawa decomposition (3.4), each element $g \in G(\mathbb{A})$ can be written as a product $g = n(b)m(a)k$. For any $g = n(b)m(a)k \in G(\mathbb{A})$, we set

$$|a(g)| = |\det(a)|_{\mathbb{A}}.$$

One can prove that this is a well-defined function, which is left-$N(\mathbb{A})M(\mathbb{Q})$ invariant and right-K invariant.

The character χ_V (3.1) is extendable to the group $P(\mathbb{A})$ via $\chi_V\begin{pmatrix} a & b \\ 0 & (a^{-1})^t\end{pmatrix} = \chi_V(\det(a))$. For $s \in \mathbb{C}$, we denote by $I_n(s, \chi_V)$ the normalized induced representation from $P(\mathbb{A})$ to $G(\mathbb{A})$. It is called the *degenerate principal series* representation. It is realized on the space of all smooth functions $\Phi(g, s)$ on $G(\mathbb{A})$ satisfying

$$\Phi(n(b)m(a)g, s) = \chi_V(\det a)|\det(a)|^{s+\rho_n}\Phi(g, s) \tag{3.18}$$

for all $a \in \mathrm{GL}_n(\mathbb{A})$ and $b \in \mathrm{Sym}_n(\mathbb{A})$, where

$$\rho_n = \frac{n+1}{2}. \tag{3.19}$$

The action of $G(\mathbb{A})$ is given by right translations. Let $I_{n,p}(s, \chi_V)$ denote the corresponding local induced representations. According to [Ku1], $I_n(s, \chi_V)$ can be written as a restricted tensor product $I_n(s, \chi_V) = \otimes_{p \leqslant \infty} I_{n,p}(s, \chi_V)$. Also, each $\varphi \in I_n(s, \chi_V)$ is determined by its restriction to K.

An element $\Phi \in I_n(s, \chi_V)$ is called a *standard section* if its restriction to K is independent of s. It is called *factorizable* if it has the form $\Phi(g, s) = \otimes_{p \leqslant \infty} \Phi_p(g, s)$ with $\Phi_p(g, s) \in I_{n,p}(s, \chi_V)$.

For this paper, the normalized standard section plays an important role. The non-archimedean components are constructed by means of the intertwining map

$$\lambda : S(V(\mathbb{A})^n) \longrightarrow I_n(s_0, \chi_V), \qquad \varphi \mapsto \Phi(g, s_0) = (\omega(g)\varphi)(0), \tag{3.20}$$

where

$$s_0 = \frac{m}{2} - \rho_n \text{ and } \omega \text{ is the Weil representation in Section 3.2.}$$

Using the Iwasawa decomposition, it can be proved that $\lambda(\varphi) \in I_n(s_0, \chi_V)$ has a unique extension to a standard section by

$$\lambda(\varphi)(g, s) = |a(g)|^{s-s_0}(\omega(g)\varphi)(0) \in I_n(s, \chi_V). \tag{3.21}$$

We now want to define a section Φ^l, which depends on a lattice L. To this end, we briefly recall some necessary facts on lattices and discriminant forms. For more details, see for example [MM].

Remark 3.2. Let $L \subset V$ be a lattice as introduced in Section 3.1. It is well known that we can attach to each prime p a p-adic lattice $L_p = L \otimes \mathbb{Z}_p$. The bilinear form $(\cdot, \cdot)$ and the corresponding quadratic form q can be extended to L_p in the usual way. It is then possible to define the dual lattice L_p' and the discriminant group L_p'/L_p, which can be endowed with a finite quadratic form q_{L_p} coming from the quadratic form on L_p. The groups L'/L and L_p'/L_p and the quadratic forms on them are closely related:

$$L_p'/L_p \cong (L'/L)_p,$$

where $(L'/L)_p$ is the p-component of L'/L. The quadratic form q_{L_p} is just the restriction of q_L to the p-component $(L'/L)_p$. In particular, for an element $\mu \in L'/L$, the p-part of μ can be interpreted as an element of L_p'/L_p.

DEFINITION 3.3. For $\mu \in (L'/L)^n$, we define $\varphi_\mu \in S(V(\mathbb{A}_f)^n)$ by

$$\varphi_\mu = \chi_{\mu+\hat{L}^n} = \prod_{p<\infty} \varphi_p^{(\mu)} = \prod_{p<\infty} \chi_{\mu+L_p^n}. \tag{3.22}$$

Here $L_p = L \otimes \mathbb{Z}_p$, which is the p-part of $\hat{L} = L \otimes \hat{\mathbb{Z}}$ with $\hat{\mathbb{Z}} = \prod_{p<\infty} \mathbb{Z}_p$, and $\chi_{\mu+L_p^n}$ is the characteristic function of $\mu + L_p^n$.

We associate to $(\varphi_p^{(\mu)})_{p<\infty}$ the standard section $\Phi_\mu^l(s) = \Phi_\infty^l(s) \prod_{p<\infty} \Phi_p^{(\mu)}(s) \in I_n(s, \chi_V)$ with $l \in \mathbb{Z}$ and

$$\begin{aligned} &\Phi_\infty^l(k, s) = \det(\mathbf{k})^l \quad \text{for all } k \in K_\infty, \\ &\Phi_p^{(\mu)}(g, s) = \lambda_p(\varphi_p^{(\mu)})(g, s), \end{aligned} \tag{3.23}$$

where $\mathbf{k} \in U(n)$ corresponds to $k \in K_\infty$.

The lemma below is well known for $n = 1$; see for example [Ho]. Although it might be known for a general n, I was not able to find it in the literature.

LEMMA 3.4. *Let $\mu = (\mu_1, \ldots, \mu_n) \in (L'/L)^n$ and $\Phi_\mu^l(s) \in I_n(s, \chi_V)$ be as in Definition 3.3. Then*

$$\Phi_p^{(\mu)}(k, s) = 1 \quad \textit{for all } k \in \mathrm{Sp}(n, \mathbb{Z}_p) \tag{3.24}$$

for all primes p coprime to $|L'/L|$. In particular, $\chi_{V,p} = (\cdot, (-1)^{m/2}\det(V))_p$ is unramified for all those primes.

Proof. The latter assumption follows directly from the explicit formulas of the local Hilbert symbol $(\cdot, (-1)^{m/2}\det(V))_p$. Note that $(-1)^{m/2}\det(V) \equiv 0, 1 \bmod 4$.

By definition, $\Phi_p^{(\mu)}(k) = (\omega(k)\varphi_p^{(\mu)})(0)$. It suffices to prove (3.24) on the generators of $\mathrm{Sp}(n, \mathbb{Z}_p)$. According to Lemma 3.1, $\mathrm{Sp}(n, \mathbb{Z}_p)$ is generated by the elements w, $n(b)$, and $m(a)$. Using the formulas (3.12), we obtain the following formulas for the Weil representation acting on $\varphi_p^{(\mu)}$, which are a straightforward generalization of [BY, (2.33)] (see also [N, Section 3.5]).

$$\begin{aligned} \Phi_p^{(\mu)}(n(b)) &= (\omega(n(b))\varphi_p^{(\mu)})(0) \\ &= \psi_p(\mathrm{tr}(bQ[\mu]))\varphi_p^{(\mu)}(0) \\ &= \psi_p(\mathrm{tr}(bQ[\mu]))\Phi_p^{(\mu)}(1) \quad \text{for } b \in \mathrm{Sym}_n(\mathbb{Z}_p), \qquad (3.25) \\ \Phi_p^{(\mu)}(w) &= (\omega(w)\varphi_p^{(\mu)})(0) \\ &= \gamma_p(V)^n |L_p'/L_p|^{-n/2} \sum_{\nu \in (L_p'/L_p)^n} \psi_p(\mathrm{tr}(\nu, \mu))\varphi_p^{(\nu)}(0) \\ &= \gamma_p(V)^n |L_p'/L_p|^{-n/2} \sum_{\nu \in (L_p'/L_p)^n} \psi_p(\mathrm{tr}(\nu, \mu))\Phi_p^{(\nu)}(1), \qquad (3.26) \end{aligned}$$

and

$$\begin{aligned} \Phi_p^{(\mu)}(m(a)) &= (\omega(m(a))\varphi_p^{(\mu)})(0) \\ &= \chi_{V,p}(\det a)|\det a|_p^{m/2}\varphi_p^{(\mu a^{-1})}(0) \\ &= \chi_{V,p}(\det a)|\det a|_p^{m/2}\Phi_p^{(\mu a^{-1})}(1) \quad \text{for } a \in \mathrm{GL}_n(\mathbb{Z}_p). \qquad (3.27) \end{aligned}$$

If p is coprime to $|L'/L|$, we can conclude the following:

(1) L_p is unimodular since $|L'_p/L_p| = p^{\mathrm{ord}_p(|L'/L|)}$; see for example [Ki, Section 5.1].
(2) $\psi_p(\mathrm{tr}\, bQ[\mu]) = \psi_p(\mathrm{tr}(\mu,\nu)) = 1$.
(3) $\gamma(V) = 1$ by [We, Theorem 5].
(4) $\chi_{V,p}(\det a) = 1$ for all $a \in \mathrm{GL}_n(\mathbb{Z}_p)$ using the definition of the Hilbert symbol.

This finally proves the claim. ∎

As in [BY], we consider the $|L'/L|^n$-dimensional subspace

$$S_L = \bigoplus_{\mu \in (L'/L)^n} \mathbb{C}\varphi_\mu \subset S(V(\mathbb{A}_f)^n). \tag{3.28}$$

From the proof of Lemma 3.4, it is obvious that space (3.28) is stable under the action of the group $\mathrm{Sp}(n, \hat{\mathbb{Z}})$ via the Weil representation. Thereby, a finite dimensional representation $\rho_{L,n}$ of $\mathrm{Sp}(n,\mathbb{Z})$ on S_L is defined by

$$\rho_{L,n}(g)\varphi = \overline{\omega}(g_f)\varphi, \tag{3.29}$$

where $g_f \in \mathrm{Sp}(n,\hat{\mathbb{Z}})$ is the projection of $g \in \mathrm{Sp}(n,\mathbb{Z})$ into $\mathrm{Sp}(n,\hat{\mathbb{Z}})$. Equations (3.25), (3.26), and (3.27) yield explicit formulas for $\rho_{L,n}$ on the generators of $\mathrm{Sp}(n,\mathbb{Z})$:

$$\begin{aligned}
\rho_{L,n}(n(b))\varphi_\mu &= e(\mathrm{tr}(bQ[\mu]))\varphi_\mu \\
\rho_{L,n}(w)\varphi_\mu &= \frac{e\left(\frac{n(b^+-b^-)}{8}\right)}{|L'/L|^{n/2}} \sum_{\nu \in (L'/L)^n} e(\mathrm{tr}(\nu,\mu))\varphi_\nu \\
\rho_{L,n}(m(a))\varphi_\mu &= \chi_V(\det a)|\det a|^{m/2}\varphi_{\mu a^{-1}}.
\end{aligned} \tag{3.30}$$

Let $\{\mathfrak{e}_\mu \mid \mu \in (L'/L)^n\}$ be the standard basis of the group ring $\mathbb{C}[(L'/L)^n]$. Then the map $\varphi_\mu \mapsto \mathfrak{e}_\mu$ defines an isomorphism between S_L and $\mathbb{C}[(L'/L)^n]$, and $\rho_{L,n}$ can be interpreted as a representation acting on the group ring $\mathbb{C}[(L'/L)^n]$. The formulas in (3.30) can also be found in [Zh, Section 2.1.3].

For later purposes, we introduce the standard inner product on $\mathbb{C}[(L'/L)^n]$ by

$$\left\langle \sum_{\lambda \in (L'/L)^n} a_\lambda \mathfrak{e}_\lambda, \sum_{\mu \in (L'/L)^n} b_\mu \mathfrak{e}_\mu \right\rangle = \sum_{\lambda \in (L'/L)^n} a_\lambda \overline{b_\lambda}. \tag{3.31}$$

Since $\rho_{L,n}$ is a unitary representation with respect to (3.31), its dual representation $\rho^*_{L,n}$ is just the complex conjugate of $\rho_{L,n}$. That is,

$$\rho^*_{L,n}(g) = \rho^{-1}_{L,n}(g) = \overline{\rho_{L,n}(g)}$$

for all $g \in \mathrm{Sp}(n,\mathbb{Z})$. The explicit formulas (3.30) show immediately that $\rho^*_{L,n}$ is equal to $\rho_{L(-1),n}$, where $L(-1)$ is the lattice L equipped with the bilinear form $-(\cdot,\cdot)$. Thus, without loss of generality, one can work with the dual Weil representation.

3.4 Local Whittaker functions

In this section, we compute a particular p-adic integral. It will turn out that a local Whittaker integral at a ramified prime can be expressed in terms of this integral. We follow mostly [KY, 4.1], [Ya, Chapter 2], and [HS]. First, we consider anisotropic discriminant forms, which will be needed in the sequel.

3.4.1 Anisotropic quadratic modules

DEFINITION 3.5. Let $(L'/L,(\cdot,\cdot))$ be a finite quadratic module with associated quadratic form q. We call $(L'/L,(\cdot,\cdot))$ *anisotropic* if $q(\mu)\neq 0$ for all $\mu\in L'/L\setminus\{0\}$.

The structure of anisotropic finite quadratic modules of prime-power order can be found in [BEF, Lemma 4.9].

REMARK 3.6. Let p be an odd prime, and let $(L'/L,(\cdot,\cdot))$ be an anisotropic finite quadratic module whose order is a power of p. Then we have

$$(L'/L,q)\cong\begin{cases}\left(\mathbb{Z}/p\mathbb{Z},q(x)=\frac{\alpha_1x^2}{p}\right)\text{ or }\left(\mathbb{Z}/p\mathbb{Z},q(x)=\frac{\alpha_2x^2}{p}\right)\\ \quad\oplus\left(\mathbb{Z}/p\mathbb{Z},q(x)=\frac{x^2}{p}\right), & p\equiv 1\bmod 4,\\ \left(\mathbb{Z}/p\mathbb{Z},q(x)=\frac{\beta_1x^2}{p}\right)\text{ or }\left(\mathbb{Z}/p\mathbb{Z},q(x)=\frac{\beta_2x^2}{p}\right)\\ \quad\oplus\left(\mathbb{Z}/p\mathbb{Z},q(x)=\frac{x^2}{p}\right), & p\equiv 3\bmod 4,\end{cases}\tag{3.32}$$

where $(\frac{\alpha_1}{p})=(\frac{\beta_1}{p})=\pm(\frac{2}{p})$, $(\frac{\beta_2}{p})=(\frac{2}{p})$, and $(\frac{\alpha_2}{p})=-(\frac{2}{p})$.

These results can be transferred to p-adic lattices.

REMARK 3.7.

(1) Let p be an odd prime, and let $(L,(\cdot,\cdot))$ be a non-degenerate and even p-adic lattice of rank m with quadratic form q (even means that $q(x)\in\mathbb{Z}_p$ for all $x\in L$). Then L'/L is a finite p-group, and $(L'/L,(\cdot,\cdot))$ can be interpreted as a finite quadratic module.
(2) If $(L'/L,(\cdot,\cdot))$ is anisotropic, the quadratic form on the lattice L is $\mathbb{Z}_p$-equivalent to

$$\begin{cases}\sum_{i=1}^{m-1}\frac{x_i^2}{2}+\frac{p\alpha_mx_m^2}{2}\text{ or }\sum_{i=1}^{m-2}\frac{x_i^2}{2}+\frac{p\delta_{m-1}x_{m-1}^2}{2}+\frac{px_m^2}{2}, & p\equiv 1\bmod 4,\\ \sum_{i=1}^{m-1}\frac{x_i^2}{2}+\frac{p\beta_mx_m^2}{2}\text{ or }\sum_{i=1}^{m-2}\frac{x_i^2}{2}+\frac{p\varepsilon_{m-1}x_{m-1}^2}{2}+\frac{px_m^2}{2}, & p\equiv 3\bmod 4,\end{cases}\tag{3.33}$$

where $(\frac{\alpha_m}{p})=(\frac{\beta_m}{p})=\pm(\frac{2}{p})$, $(\frac{\delta_{m-1}}{p})=-(\frac{2}{p})$, and $(\frac{\varepsilon_{m-1}}{p})=(\frac{2}{p})$ (see [MM, Proposition 2.12], applied to (3.32)).

3.4.2 Local Whittaker integrals

LEMMA 3.8. *Let $p>2$ be a prime, $l\in\mathbb{Z}$ and let $(L,(\cdot,\cdot))$ be an even and non-degenerate p-adic lattice with associated quadratic form q such that the attached discriminant form L'/L is anisotropic. Further, let $V=L\otimes\mathbb{Q}_p$, d_Vx the self-dual Haar measure on V, dx the standard Haar measure on $\mathbb{Z}_p^m$, and $\varepsilon\in\mathbb{Z}_p^\times$.*

If $L'/L \cong \mathbb{Z}/p\mathbb{Z}$, we have

$$\int_L \psi_p(\varepsilon p^{-l} q(x)) d_V x = |L'/L|^{-1/2} \begin{cases} 1 & \text{if } l \leqslant 0, \\ \left(\left(\frac{-\varepsilon}{p}\right)\delta_p\right)^{m-1} p^{-(1/2)(ml-1)} & \text{if } l \geqslant 1 \text{ is odd}, \\ \pm \left(\frac{2}{p}\right)\left(\frac{-\varepsilon}{p}\right)\delta_p p^{-(1/2)(ml-1)} & \text{if } l > 1 \text{ is even} \end{cases} \tag{3.34}$$

and

$$\int_{L\times L} \psi_p(\varepsilon p^{-l}(x,y))\, d_V x\, d_V y = |L'/L|^{-1} \cdot \begin{cases} p^{-lm+1} & \text{if } l \geqslant 1, \\ 1 & \text{if } l < 1. \end{cases}$$

Similarly, if $L'/L \cong (\mathbb{Z}/p\mathbb{Z})^2$,

$$\int_L \psi_p(\varepsilon p^{-l} q(x)) d_V x = |L'/L|^{-1/2} \begin{cases} 1 & \text{if } l \leqslant 0, \\ \left(\frac{-1}{p}\right)^{(m-2)/2} p^{-(1/2)(ml-2)} & \text{if } l \geqslant 1 \text{ is odd}, \\ \left(\frac{-1}{p}\right)(-1)^{(p-3)/2}\left(\frac{2}{p}\right) p^{-(1/2)(ml-2)} & \text{if } l > 1 \text{ is even} \end{cases} \tag{3.35}$$

and

$$\int_{L\times L} \psi_p(\varepsilon p^{-l}(x,y))\, d_V x\, d_V y = |L'/L|^{-1} \cdot \begin{cases} p^{-lm+2} & \text{if } l \geqslant 1, \\ 1 & \text{if } l < 1. \end{cases}$$

As usual,

$$\delta_p = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4, \\ i & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

Proof. By [KY, p. 2287], we have $d_V x = |L'/L|^{-1/2}\, dx$. Also, by Remark 3.7, q is $\mathbb{Z}_p$-equivalent to one of the quadratic forms in (3.33).

The integral $\int_{\mathbb{Z}_p^m} \psi_p(\varepsilon p^{-l} q(x)) dx$ is computed explicitly for a general quadratic form in [Ya, Lemma 2.2]. The first formula of (3.34) and (3.35) respectively follows easily from the general formula applied to the special quadratic form.

The integral $\int_{\mathbb{Z}_p^m \times \mathbb{Z}_p^m} \psi_p(\varepsilon p^{-l}(x,y))\, dx\, dy$ is calculated in [HS, Proposition 3.2], for a general quadratic form. The second formula of (3.34) and (3.35) respectively follows immediately from the just quoted general one. □

For the following lemma, we introduce some notation, which is adopted from [HS]. We consider the set

$$T_n(\mathbb{Q}_p) = \{b \in \mathrm{Sym}_n(\mathbb{Q}_p) \mid \det(b) \neq 0\}$$

and the subgroup

$$\Gamma_0(p) = \{\gamma = (\gamma_{ij}) \in \mathrm{GL}_n(\mathbb{Z}_p) \mid \gamma_{ij} \equiv 0 \bmod p \text{ for all } i > j\}$$

of $\mathrm{GL}_n(\mathbb{Z}_p)$. It acts on $T_n(\mathbb{Q}_p)$ by $b \mapsto \gamma \cdot b = \gamma b \gamma^t$. For the next lemma, we will need a complete set of representatives of $\Gamma_0(p)\backslash T_n(\mathbb{Q}_p)$ with respect to this action, which was determined in [HS, Theorem 2.1]. To describe this result, let S_n be the symmetric group acting on $I = \{1, \ldots, n\}$, and let $\delta \in \mathbb{Z}_p^\times \setminus (\mathbb{Z}_p^\times)^2$ be a p-adic non-square unit. Such a set of representatives is given by

$$\begin{aligned} &\{S_{\sigma,e,\varepsilon} = (s_{ij}) \mid s_{ij} = \varepsilon_i p^{e_i} \delta_{i,\sigma(j)} \\ &\quad \text{with } (\sigma, e = (e_1, \ldots, e_n), \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)) \in S_n \times \mathbb{Z}^n \times \{1, \delta\}^n\}, \end{aligned} \tag{3.36}$$

where

$$\sigma^2 = \mathrm{id}, \qquad e_{\sigma(i)} = e_i \quad \text{for all } i \in I, \qquad \varepsilon_i = 1 \quad \text{if } \sigma(i) \neq i. \tag{3.37}$$

The following constants occur in the formula for local densities in [HS, Theorem 4.1]. For the convenience of the reader, we repeat these quantities in full detail.

$$\begin{aligned} c_1(\sigma) &= |\{i \in I \mid \sigma(i) = i, \}| \qquad \text{and} \\ c_2(\sigma) &= |\{i \in I \mid \sigma(i) \neq i, \}|. \end{aligned}$$

For a partition $I = I_0 \cup \cdots \cup I_s$ into disjoint σ-stable subsets, we put

$$\begin{aligned} n_l &= |I_l|, \\ n(l) &= \frac{n^{(l)}(n^{(l)}+1)}{2}, \end{aligned} \tag{3.38}$$

where $n^{(l)} = n_l + n_{l+1} + \cdots + n_s$ and $l \geqslant 0$. We also define

$$\begin{aligned} t(\sigma, \{I_i\}) &= \sum_{l=0}^{s} |\{(i,j) \in I_l \times I_l \mid i < j < \sigma(i), \sigma(j) < \sigma(i)\}|, \\ \tau(\{I_i\}) &= \sum_{l=1}^{s} |\{(i,j) \in I_l \times (I_0 \cup \cdots \cup I_{l-1}) \mid j < i\}|, \\ c_1^{(l)}(\sigma) &= |\{i \in I_l \cup \cdots \cup I_s \mid \sigma(i) = i\}|, l \geqslant 0, \\ &\text{and} \\ c_{1,l}(\sigma) &= |\{i \in I_l \mid \sigma(i) = i\}|, \\ c_{2,l}(\sigma) &= |\{i \in I_l \mid \sigma(i) > i\}|. \end{aligned} \tag{3.39}$$

The terms $c_{1,l}(\sigma)$ and $c_{2,l}(\sigma)$ occur frequently in subsequent formulas in this paper. In order to make these formulas more readable, we drop the (σ)-part and write from now on $c_{1,l}$ and $c_{2,l}$ instead.

Let $(L, (\cdot,\cdot))$ be a p-adic lattice as in Lemma 3.8, and let q be the attached quadratic form. For $r \in \mathbb{N}$, we set $L_r = L \oplus \mathbb{Z}_p^{2r}$, let q_r be the quadratic form

$$q_r = q + q' \quad \text{with } q'(x) = \frac{1}{2} x^t \begin{pmatrix} 0 & 1_r \\ 1_r & 0 \end{pmatrix} x, \tag{3.40}$$

and denote the corresponding bilinear form by $(\cdot,\cdot)_r$. We then define for $\varepsilon \in \mathbb{Z}_p^\times$ and $l \in \mathbb{Z}$

$$J_r(\varepsilon p^l) = \int_{L_r} \psi_p(\varepsilon p^l q_r(x))\, d_V x \qquad \text{and} \qquad I_r(\varepsilon p^l) = \int_{L_r \times L_r} \psi_p(2\varepsilon p^l (x,y)_r)\, d_V x\, d_V y.$$

Finally, let $k \in \mathbb{N}$. The following set will be used later on:

$$\{(\nu_0, \nu_1, \ldots, \nu_{k-1}) \in \mathbb{Z} \times \mathbb{N}^{k-1} \mid \nu_0 + \nu_1 + \cdots + \nu_l \leqslant -1, 0 \leqslant l \leqslant k-1\}. \tag{3.41}$$

The variable ν_0 can be parametrized by $t \in \mathbb{N}$ by setting $\nu_0 = -t - \sum_{i=1}^{k-1} \nu_i$. Note that t is determined by a tuple ν by the previously stated equation for ν_0. Based on this parametrization, (3.41) can be written in the form

$$\{\nu\}_k = \left\{ \left(\nu_0 = -\sum_{i=1}^{k-1} \nu_i - t, \nu_1, \ldots, \nu_{k-1} \right) \in \mathbb{Z} \times \mathbb{N}^{k-1} \mid t \in \mathbb{N} \right\}, \tag{3.42}$$

which is more suitable for later estimates and calculations. To each tuple $\nu = (\nu_0 = -\sum_{i=1}^{k-1} \nu_i - t, \nu_1, \ldots, \nu_{k-1})$, we associate the quantities

$$\begin{aligned} S_l^{(k)}(\nu) &= \sum_{i=0}^{l} \nu_i = -t - \sum_{i=l+1}^{k-1} \nu_i, \quad 0 \leqslant l \leqslant k-1, \\ o(\nu)_k &= \{0 \leqslant l \leqslant k-1 \mid S_l^{(k)}(\nu) \equiv 1 \bmod 2\}, \\ e(\nu)_k &= \{0 \leqslant l \leqslant k-1 \mid S_l^{(k)}(\nu) \equiv 0 \bmod 2\} \end{aligned} \tag{3.43}$$

and

$$\Xi(\sigma, \nu, q_r) = \begin{cases} 0 & \text{if } L'/L \cong \mathbb{Z}/p\mathbb{Z}, \\ (-1)^{\sum_{l \in e(\nu)_k} c_{1,l}} \left(\dfrac{-1}{p} \right)^{(m-2)/2 \sum_{l \in o(\nu)_k} c_{1,l}} & \\ \quad \times \left(\dfrac{2}{p} \right)^{\sum_{l \in e(\nu)_k} c_{1,l}} p^{(m+2r) \sum_{l=0}^{k-1} S_l^{(k)}(\nu)((1/2)c_{1,l} + c_{2,l})} & \text{if } L'/L \cong (\mathbb{Z}/p\mathbb{Z})^2. \end{cases}$$

Note that the sets $o(\nu)_k$ and $e(\nu)_k$ depend only on the parity of each component ν_i. In this context, we further introduce the subsets

$$\{\nu_{i_1, \ldots, i_d}\}_k, \quad 0 \leqslant i_1 < i_2 < \cdots < i_d \leqslant n, \tag{3.44}$$

of $\{\nu\}_k$, which consist of all tuples ν, where the components ν_{i_j}, $j = 1, \ldots, d$, run over all odd integers and the remaining components over all even integers. Note that if the parity of $\nu_1, \ldots, \nu_{k-1}$ is fixed, then the parity of ν_0 is determined by that of t. To emphasize the dependency of $o(\nu)_k$ and $e(\nu)_k$ on the parity of the components of ν, we will later write $o(i_1, \ldots, i_d)_k$ and $e(i_1, \ldots, i_d)_k$ respectively if $\nu \in \{\nu_{i_1, \ldots, i_d}\}_k$.

LEMMA 3.9. *Let $p > 2$ be a prime, $(L, (\cdot, \cdot))$ be a p-adic lattice of even rank m with the same properties as in Lemma 3.8, and $\mu \in L_r'/L_r$ for $r \in \mathbb{N}$.*

Then

$$\begin{aligned} &\int_{\mathrm{Sym}_n(\mathbb{Q}_p)} \int_{\mu + L_r^n} \psi_p \left(\frac{1}{2} \operatorname{tr}(bQ^r[x]) \right) dx\, db \\ &= \delta_{\mu,0} \frac{1}{|L'/L|^{n/2}} \sum_{\substack{\sigma \in S_n \\ \sigma^2 = \mathrm{id}}} 2^{-c_1(\sigma)} (1 - p^{-1})^{c_2(\sigma) + c_1(\sigma)} p^{-c_2(\sigma)} \end{aligned}$$

$$\times \sum_{I=I_0\cup\cdots\cup I_s} p^{-\tau(\{I_i\})-t(\sigma,\{I_i\})}$$
$$\times \sum_{k=0}^{s+1} \frac{2^{c_1^{(k)}(\sigma)} 2^{\sum_{l=0}^{k-1} c_{1,l}} (1-p^{-1})^{c_1^{(k)}(\sigma)} p^{-\sum_{l=k+1}^{s} n(l)} p^{\sum_{l=0}^{k-1}(c_{1,l}+2c_{2,l})}}{\prod_{l=k}^{s}(1-p^{-n(l)})}$$
$$\times \sum_{\{\nu\}_k} p^{\sum_{j=0}^{k-1}\nu_j(n(k)-n(j))} \Xi(\sigma,\nu,q_r), \tag{3.45}$$

where

(1) *$Q^r[x]$ is the moment matrix with respect to $(\cdot,\cdot)_r$;*
(2) *dx is the self-dual Haar measure on $L_r^n \otimes \mathbb{Q}_p$ with respect to the character ψ_p;*
(3) *db is the product measure of the standard Haar measure on $\mathbb{Q}_p$;*
(4) *$\delta_{\mu,0}$ is the Kronecker delta and $s \leqslant n$ is an integer;*
(5) *$\sum_{I=I_0\cup\cdots\cup I_s}$ is the sum over all partitions of I into disjoint σ-stable subsets;*
(6) *and the summation with respect to $\{\nu\}_k$ is taken over the set (3.42); if $k=0$, we understand this sum to be equal to 1.*

Proof. First, note that by an idea of Liu [Li2, Proposition 2.4], for $r > \rho_n$, in the region of absolute convergence, we have

$$\begin{aligned}\int_{\mathrm{Sym}_n(\mathbb{Q}_p)} \int_{\mu+L_r^n} \psi_p\left(\frac{1}{2}\operatorname{tr}(bQ^r[x])\right) dx\, db &= \int_{\mu+L_r^n} \int_{\mathrm{Sym}_n(\mathbb{Q}_p)} \psi_p\left(\frac{1}{2}\operatorname{tr}(bQ^r[x])\right) db\, dx \\ = \int_{\mu+L_r^n} \int_{\mathrm{Sym}_n(\mathbb{Q}_p)/\mathrm{Sym}_n(\mathbb{Z}_p)} &\psi_p\left(\frac{1}{2}\operatorname{tr}(bQ^r[x])\right) db \\ \times \int_{b'\in\mathrm{Sym}_n(\mathbb{Z}_p)} &\psi_p\left(\frac{1}{2}\operatorname{tr}(b'Q^r[x])\right) db'\, dx.\end{aligned}$$

A simple calculation applying standard p-adic integration formulas as in [FGKP, Example 3.15] allows us to conclude that the integral over b' is equal to one if and only if

$$q_r(x_i) \in \mathbb{Z}_p \qquad \text{and} \qquad (x_i,x_j)_r \in \mathbb{Z}_p$$

for all $i,j = 1,\ldots,n$ and zero otherwise. Since L'/L is anisotropic, this is the case if and only if $\mu_1,\ldots,\mu_n \in L$. The double integral on the left-hand side in (3.45) then simplifies to

$$\int_{\mathrm{Sym}_n(\mathbb{Q}_p)} \int_{L_r^n} \psi_p\left(\frac{1}{2}\operatorname{tr}(bQ^r[x])\right) dx\, db. \tag{3.46}$$

It can be evaluated using the methods in [HS], where the local density $\alpha_p(S,T)$ depending on the matrices $S = \operatorname{diag}(u_1p^{\alpha_1},\ldots,u_mp^{\alpha_m})$ and $T = \operatorname{diag}(v_1p^{\beta_1},\ldots,v_np^{\beta_n})$, $u_i, v_j \in \mathbb{Z}_p^\times$, $\alpha_i, \beta_j \in \mathbb{Z}$, is calculated for an odd prime p and $n \in \mathbb{N}$. By comparing (3.46) with the integral in [HS, Lemma 1.1], it becomes apparent that we are interested in the case $T=0$. It can be verified that all results from [HS] are valid for $T=0$, that is, for $\beta_1 = \cdots = \beta_n = \infty$. In this case, the explicit formula for the integral $\mathscr{G}_{\Gamma_0(p)}(S_{\sigma,e,\varepsilon},T)$ in [HS, Proposition 3.3] is equal to

$$(1-p^{-1})^{2c_2(\sigma)+c_1(\sigma)} p^{-n(n-1)/2}.$$

In particular, condition (9) in this proposition is always satisfied and $\mathscr{G}_{\Gamma_0(p)}(S_{\sigma,e,\varepsilon},T)$ is nonzero for all $e\in\mathbb{Z}^n$. Also, the bound $b_l(\sigma,T)$ in the introduction of [HS] is equal to ∞ for all $l\in\mathbb{N}$ and $\sigma\in S_n$. Consequently, the sum $\sum_{i=0}^{l}\nu_i$ in the set

$$\{(\nu_0,\nu_1,\ldots,\nu_{k-1})\in\mathbb{Z}\times\mathbb{N}^{k-1}\mid -b_l(\sigma,T)\leqslant\nu_0+\nu_1+\cdots+\nu_l\leqslant -1, 0\leqslant l\leqslant k-1\} \tag{3.47}$$

appearing in [HS, Theorem 4.1] is not bounded from below for $T=0$. Note that (without the bound $-b_l(\sigma,T)$) it is equal to $\{\nu\}_k$ in (3.42).

Since [HS, Proposition 1.2] can also be carried over to our situation, we obtain

$$\begin{aligned}&\int_{\mathrm{Sym}_n(\mathbb{Q}_p)}\int_{L_r^n}\psi_p\left(\frac{1}{2}\operatorname{tr}(bQ^r[x])\right)dx\,db\\ &\quad=p^{-n(n-1)/2}\sum_{b\in\Gamma_0(p)\backslash T_n(\mathbb{Q}_p)}\frac{(1-p^{-1})^{2c_2(\sigma)+c_1(\sigma)}}{\alpha(\Gamma_0(p),b)}\int_{L_r^n}\psi_p\left(\frac{1}{2}\operatorname{tr}(bQ^r[x])\right)dx.\end{aligned} \tag{3.48}$$

The term $\alpha(\Gamma_0(p),b)$ is explicitly calculated in [HS, Theorem 2.2]: for some representative $S_{\sigma,e,\varepsilon}$ of $\Gamma_0(p)\backslash T_n(\mathbb{Q}_p)$, the expression $\alpha(\Gamma_0(p),S_{\sigma,e,\varepsilon})$ is equal to

$$\alpha(\Gamma_0(p),S_{\sigma,e,\varepsilon})=2^{c_1(\sigma)}(1-p^{-1})^{c_2(\sigma)}p^{c(\sigma,e,\varepsilon)}, \tag{3.49}$$

where

$$c(\sigma,e,\varepsilon)=-\frac{n(n-1)}{2}+\tau(\{I_i\})+t(\sigma,\{I_i\})+c_2(\sigma)+\sum_{l=0}^{s}\nu_l n(l).$$

The same ideas for calculating the integral on the right-hand side of (3.48), as in [HS, Proposition 3.2], lead to

$$\begin{aligned}&\sum_{b\in\Gamma_0(p)\backslash T_n(\mathbb{Q}_p)}\frac{1}{\alpha(\Gamma_0(p),b)}\int_{L_r^n}\psi_p\left(\frac{1}{2}\operatorname{tr}(bQ^r[x])\right)dx\\ &\quad=\sum_{(\sigma,e,\varepsilon)}2^{-c_1(\sigma)}(1-p^{-1})^{-c_2(\sigma)}p^{-c(\sigma,e,\varepsilon)}\\ &\qquad\times\prod_{\substack{i=1\\ \sigma(i)=i}}^{n}\int_{L_r}\psi_p(\varepsilon_i p^{e_i}q_r(x))\,d_Vx\cdot\prod_{\substack{j=1\\ \sigma(j)>j}}^{n}\int_{L_r\times L_r}\psi_p(2\varepsilon_j p^{e_j}(x,y)_r)\,d_Vx\,d_Vy.\end{aligned}$$

Following the proof of Theorem 4.1 of [HS] and taking the thoughts with respect to set (3.47) from above into account, we find that the last expression is equal to

$$\begin{aligned}&p^{n(n-1)/2}\sum_{\substack{\sigma\in S^n\\ \sigma^2=\mathrm{id}}}2^{-c_1(\sigma)}(1-p^{-1})^{-c_2(\sigma)}p^{-c_2(\sigma)}\sum_{I=I_0\cup\cdots\cup I_s}p^{-\tau(\{I_i\})-t(\sigma,\{I_i\})}\\ &\quad\times\sum_{k=0}^{s+1}\frac{2^{c_1^{(k)}(\sigma)}(1-p^{-1})^{c_1^{(k)}(\sigma)}p^{-\sum_{l=k+1}^{s}n(l)}}{\prod_{l=k}^{s}(1-p^{-n(l)})}\\ &\quad\times\sum_{\{\nu\}_k}p^{\sum_{j=0}^{k-1}\nu_l(n(k)-n(j))}\prod_{l=0}^{k-1}\prod_{\substack{i\in I_l\\ \sigma(i)=i}}\left(\sum_{\varepsilon_i\in\{1,\delta\}}J_r(\varepsilon_i p^{S_l^{(k)}(\nu)})\right)\prod_{\substack{i\in I_l\\ \sigma(i)>i}}I_r(2p^{S_l^{(k)}(\nu)}).\end{aligned}$$

In light of [Ya, Corollary 2.3 and Notation (1.6)], we obtain

$$\begin{aligned}&\prod_{l=0}^{k-1}\prod_{\substack{i\in I_l\\ \sigma(i)=i}}\left(\sum_{\varepsilon_i\in\{1,\delta\}}J_r(\varepsilon_i p^{S_l^{(k)}(\nu)})\right)\prod_{\substack{i\in I_l\\ \sigma(i)>i}}I_r(2p^{S_l^{(k)}(\nu)})\\ &=\prod_{l=0}^{k-1}\prod_{\substack{i\in I_l\\ \sigma(i)=i}}\left(1+\left(\frac{\delta}{p}\right)^{l(S_l^{(k)}(\nu),1)}\right)J_r(p^{S_l^{(k)}(\nu)})\prod_{\substack{i\in I_l\\ \sigma(i)>i}}I_r(2p^{S_l^{(k)}(\nu)}).\end{aligned}$$

Decomposing the product $\prod_{l=0}^{k-1}$ into products over $o(\nu)_k$ and $e(\nu)_k$ and using formulas (3.34) and (3.35) allows us to write the last expression in the form

$$\begin{aligned}&|L'/L|^{-n/2}\\ &\times\begin{cases}0 & \text{if } L'/L\cong\mathbb{Z}/p\mathbb{Z}\\ \displaystyle\prod_{l\in o(\nu)_k}\left(2\left(\frac{-1}{p}\right)^{(m-2)/2}p^{(1/2)((m+2r)S_l^{(k)}(\nu)+2)}\right)^{c_{1,l}} & \\ \displaystyle\quad\times\prod_{l\in e(\nu)_k}\left(-2\left(\frac{2}{p}\right)p^{(1/2)((m+2r)S_l^{(k)}(\nu)+2)}\right)^{c_{1,l}} & \\ \displaystyle\quad\times\prod_{l=0}^{k-1}p^{((m+2r)S_l^{(k)}(\nu)+2)c_{2,l}} & \text{if } L'/L\cong(\mathbb{Z}/p\mathbb{Z})^2,\end{cases}\end{aligned}$$

which leads to the formula in (3.45). ∎

3.5 Eisenstein series

3.5.1 Adelic and vector valued Eisenstein series

To any standard section $\Phi\in I_n(s,\chi_V)$ and $g\in G(\mathbb{A})$, we can associate an Eisenstein series

$$E(g,s,\Phi)=\sum_{\gamma\in P(\mathbb{Q})\backslash G(\mathbb{Q})}\Phi(\gamma g,s). \tag{3.50}$$

It converges absolutely for $\mathrm{Re}(s)>\rho_n$ and defines an automorphic form on $G(\mathbb{A})$. Moreover, as is important for this paper, it can be proved that $E(g,s,\Phi)$ has a meromorphic analytic continuation to the whole complex s-plane and that it satisfies a functional equation

$$E(g,s,\Phi)=E(g,-s,M(s)\Phi), \tag{3.51}$$

where

$$\begin{aligned}M(s)=\prod_{p\leqslant\infty}M_p(s):I_n(s,\chi_V)&\longrightarrow I_n(-s,\chi_V),\\ \Phi(g,s)\mapsto M(s)\Phi(g,s)&=\int_{N(\mathbb{A})}\Phi(wng,s)\,dn\end{aligned} \tag{3.52}$$

is the global intertwining operator. Since $N(\mathbb{A}) \cong \mathrm{Sym}_n(\mathbb{A})$, for each $p \leqslant \infty$, the operator $M_p(s)$ can be written as

$$M_p(s)\Phi_p(g, s) = \int_{\mathrm{Sym}_n(\mathbb{Q}_p)} \Phi_p(wn(b)g, s)\, db, \tag{3.53}$$

where $\mathbb{Q}_\infty = \mathbb{R}$ and db is the self-dual Haar measure on $\mathrm{Sym}_n(\mathbb{Q}_p)$ with respect to the pairing $(b_1, b_2) \mapsto \psi_p(\mathrm{tr}(b_1 b_2))$.

The following result is well known; see [KR, Lemma 4.6]. We give an elementary proof, which is in principal also known, but to the best of my knowledge not explicitly written down. So, we add it here for the convenience of the reader.

THEOREM 3.10. *Let $\Phi^l_\infty(s)$ be the archimedean part of the section defined in (3.23), $\alpha = \frac{1}{2}(s + \rho_n + l)$, and $\beta = \frac{1}{2}(s + \rho_n - l)$. Then*

$$M_\infty(s)\Phi^l_\infty(g, s) = i^{-ln} 2^{(1-s)n} \pi^{n(n+1)/2} \frac{\Gamma_n(s)}{\Gamma_n(\alpha)\Gamma_n(\beta)} \Phi^l_\infty(g, -s) \tag{3.54}$$

for all $g \in \mathrm{Sp}(n, \mathbb{R})$.

Proof. The proof is basically a straightforward generalization of the analogous result in [Ku2, Lemma 9.2]. Let $n(x)m(a)k$ be the Iwasawa decomposition of g. We can assume that $\det(a) > 0$. Lemma 9.2 and the subsequent arguments in [Ku2] carry over to the case of a general n. We therefore obtain (see also [BFK, (1.5)])

$$\begin{aligned} &\int_{\mathrm{Sym}_n(\mathbb{R})} \Phi^l_\infty(wn(b)n(x)m(a)k, s)\, db \\ &\quad = \chi_V(\det(a)) |\det(a)|^{s+\rho_n} \det(\mathbf{k})^l \int_{\mathrm{Sym}_n(\mathbb{R})} \det(b + iy)^{-\alpha} \det(b - iy)^{-\beta}\, db, \end{aligned} \tag{3.55}$$

where $y = a^t a$. The integral on the right-hand side of (3.55) can be evaluated with the help of [Sh, (1.31)]. We find for $\mathrm{Re}(s + \rho_n) > n + 1$

$$\int_{\mathrm{Sym}_n(\mathbb{R})} \det(b + iy)^{-\alpha} \det(b - iy)^{-\beta}\, db = i^{-ln} 2^{(1-s)n} \pi^{n(n+1)/2} \det(y)^{-s} \frac{\Gamma_n(s)}{\Gamma_n(\alpha)\Gamma_n(\beta)}. \tag{3.56}$$

Combining (3.55) and (3.56), we finally obtain

$$\begin{aligned} &\int_{\mathrm{Sym}_n(\mathbb{R})} \Phi^l_\infty(wn(b)n(b)n(x)m(a)k, s)\, db \\ &\quad = i^{-ln} 2^{(1-s)n} \pi^{n(n+1)/2} \frac{\Gamma_n(s)}{\Gamma_n(\alpha)\Gamma_n(\beta)} |\det(a)|^{-s+\rho_n} \det(\mathbf{k})^l. \end{aligned}$$ ∎

The next theorem provides the local analogue of Theorem 3.10 for an *odd* ramified prime p. We first need a preparatory lemma.

LEMMA 3.11. *Let the sets $\{\nu\}_k$ and $\{\nu_{i_1,\ldots,i_d}\}_k$ be defined as in (3.42) and (3.44) respectively, $n(l)$ be given in (3.38), $X = p^{-s}$, and*

$$v(\nu) = -\sum_{l=0}^{k-1} S^{(k)}_l(\nu)(c_{1,l} + 2c_{2,l}) \tag{3.57}$$

with $c_{1,l}, c_{2,l}$ *specified in (3.39). We further define coefficients depending on the decomposition* $I = I_0 \cup \cdots \cup I_s$ *of* I *(indicated by the abbreviation* $\mathbf{I_l}$*) by*

$$A_i(\mathbf{I_l}) = \sum_{j=0}^{i-1}\left(\frac{1}{2}c_{1,j} + c_{2,j}\right), \qquad B_i(\mathbf{I_l}) = -mA_i(\mathbf{I_l}) + n(0) - n(i),\ i = 1, \ldots, k. \tag{3.58}$$

Then the series

$$\sum_{\{\nu\}_k}(-1)^{\sum_{l\in e(\nu)_k} c_{1,l}}\left(\frac{-1}{p}\right)^{((m-2)/2)\sum_{l\in o(\nu)_k} c_{1,l}}\left(\frac{2}{p}\right)^{\sum_{l\in e(\nu)_k} c_{1,l}} \times p^{\sum_{j=0}^{k-1}\nu_j(n(k)-n(j))}p^{m\sum_{l=0}^{k-1}S_l^{(k)}(\nu)((1/2)c_{1,l}+c_{2,l})}X^{v(\nu)} \tag{3.59}$$

is equal to

$$\sum_{0\leqslant i_1\leqslant\cdots\leqslant i_d\leqslant n}(-1)^{\sum_{l\in e(i_1,\ldots,i_d)_k} c_{1,l}}\left(\frac{-1}{p}\right)^{((m-2)/2)\sum_{l\in o(i_1,\ldots,i_d)_k} c_{1,l}}\left(\frac{2}{p}\right)^{\sum_{l\in e(i_1,\ldots,i_d)_k} c_{1,l}} \times p^{\sum_{j=1}^{d} 2A_{i_j}(\mathbf{I_l})s - B_{i_j}(\mathbf{I_l})}\prod_{j=1}^{k}(\zeta_p(4A_j(\mathbf{I_l})s - 2B_j(\mathbf{I_l})) - 1),$$

where the sum $\sum_{0\leqslant i_1\leqslant\cdots\leqslant i_d\leqslant n}$ *runs over all subsets of* $\{0, \ldots, n\}$ *indicating the positions of the odd components of* ν *(see (3.44)).*

In particular, (3.59) can be meromorphically continued to the whole complex s-plane.

Proof. Series (3.59) is bounded by

$$\sum_{\{\nu\}_k}\left|p^{\sum_{j=0}^{k-1}\nu_l(n(k)-n(j))}p^{m\sum_{l=0}^{k-1}S_l^{(k)}(\nu)((1/2)c_{1,l}+c_{2,l})}X^{v(\nu)}\right|.$$

Since the exponents $m\sum_{l=0}^{k-1}S_l^{(k)}(\nu)((1/2)c_{1,l}+c_{2,l})$ are negative for all $\nu \in \{\nu\}_k$ and $n(k) - n(l)$ is negative for all $1 \leqslant l \leqslant k-1$, the latter series in turn is bounded by $\sum_{\{\nu\}_k} p^{S_0^{(k)}(\nu)(n(k)-n(0))}p^{\mathrm{Re}(s)\sum_{l=0}^{k-1}S_l^{(k)}(\nu)}$, where we have used $\nu_0 = S_0^{(k)}(\nu)$. The identity $\sum_{l=0}^{k-1}S_l^{(k)}(\nu) = -kt - \sum_{i=1}^{k-1} i\nu_i$ gives

$$\begin{aligned}\sum_{\{\nu\}_k} & p^{S_0^{(k)}(\nu)(n(k)-n(0))}p^{\mathrm{Re}(s)\sum_{l=0}^{k-1}S_l^{(k)}(\nu)} \\ &= \prod_{i=1}^{k-1}\left(\sum_{\nu_i=1}^{\infty}p^{-\nu_i(i\,\mathrm{Re}(s)+(n(k)-n(0)))}\right)\sum_{t=1}^{\infty}p^{-t(k\,\mathrm{Re}(s)+(n(k)-n(0)))} \\ &\leqslant \prod_{i=1}^{k-1}\left(\sum_{n=1}^{\infty}n^{-(i\,\mathrm{Re}(s)+(n(k)-n(0)))}\right)\left(\sum_{n=1}^{\infty}n^{-(k\,\mathrm{Re}(s)+(n(k)-n(0)))}\right).\end{aligned} \tag{3.60}$$

The right-hand side of (3.60) is uniformly convergent for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > (1 + (n(0) - n(k)))/2$, which shows that the sum over $\{\nu\}_k$ in (3.59) is holomorphic in the same region.

The absolute convergence of the sum $\sum_{\{\nu\}_k}$ allows us to change the order of summation. Since $o(\nu)_k$ and $e(\nu)_k$ only depend on the parity of the components of ν, we decompose series (3.59) into partial series in the following way:

$$\sum_{0\leqslant i_1\leqslant\cdots\leqslant i_d\leqslant n}\sum_{\nu\in\{\nu_{i_1,\ldots,i_d}\}_k}(-1)^{\sum_{l\in e(\nu)_k}c_{1,l}}\left(\frac{-1}{p}\right)^{((m-2)/2)\sum_{l\in o(\nu)_k}c_{1,l}}\left(\frac{2}{p}\right)^{\sum_{l\in e(\nu)_k}c_{1,l}}$$

$$\times p^{\sum_{j=0}^{k-1}\nu_j(n(k)-n(j))}p^{m\sum_{l=0}^{k-1}S_l^{(k)}(\nu)((1/2)c_{1,l}+c_{2,l})}X^{v(\nu)}$$

$$=\sum_{0\leqslant i_1\leqslant\cdots\leqslant i_d\leqslant n}(-1)^{\sum_{l\in e(i_1,\ldots,i_d)_k}c_{1,l}}\left(\frac{-1}{p}\right)^{((m-2)/2)\sum_{l\in o(i_1,\ldots,i_d)_k}c_{1,l}}\left(\frac{2}{p}\right)^{\sum_{l\in e(i_1,\ldots,i_d)_k}c_{1,l}}$$

$$\times\sum_{\nu\in\{\nu_{i_1,\ldots,i_d}\}_k}p^{\sum_{j=0}^{k-1}\nu_j(n(k)-n(j))}p^{m\sum_{l=0}^{k-1}S_l^{(k)}(\nu)((1/2)c_{1,l}+c_{2,l})}X^{v(\nu)}. \tag{3.61}$$

A straightforward calculation yields

$$\sum_{l=0}^{k-1}S_l^{(k)}(\nu)\left(\frac{1}{2}c_{1,l}+c_{2,l}\right)=-t\left(\sum_{l=0}^{k-1}\left(\frac{1}{2}c_{1,l}+c_{2,l}\right)\right)-\sum_{l=1}^{k-1}\left(\sum_{j=0}^{l-1}\left(\frac{1}{2}c_{1,j}+c_{2,l}\right)\right)\nu_l. \tag{3.62}$$

Moreover, by using $\nu_0=S_0^{(k)}(\nu)$, it is also easily seen that

$$\sum_{l=0}^{k-1}\nu_l(n(k)-n(l))=t(n(0)-n(k))+\sum_{l=1}^{k-1}\nu_l(n(0)-n(l)). \tag{3.63}$$

In terms of coefficients (3.58), we can then write

$$m\sum_{l=0}^{k-1}S_l^{(k)}(\nu)\left(\frac{1}{2}c_{1,l}+c_{2,l}\right)+\sum_{l=0}^{k-1}\nu_l(n(k)-n(l))+2s\sum_{l=0}^{k-1}S_l^{(k)}(\nu)\left(\frac{1}{2}c_{1,l}+c_{2,l}\right)$$

$$=tB_k(\mathbf{I}_1)+\sum_{l=1}^{k-1}\nu_lB_l(\mathbf{I}_1)-2s\left(tA_k(\mathbf{I}_1)+\sum_{l=1}^{k-1}\nu_lA_l(\mathbf{I}_1)\right).$$

Taking the latter equation into account and using the variables $\nu_{i_j}=2\mu_{i_j}-1,\ j=1,\ldots,d,\ \nu_j=2\mu_j,\ j\neq i_1,\ldots,i_d$, we obtain

$$\sum_{\nu\in\{\nu_{i_1,\ldots,i_d}\}_k}p^{\sum_{j=0}^{k-1}\nu_j(n(k)-n(j))}p^{m\sum_{l=0}^{k-1}S_l^{(k)}(\nu)((1/2)c_{1,l}+c_{2,l})}X^{v(\nu)}$$

$$=\prod_{j=1}^{d}\left(\sum_{\mu_{i_j}=1}^{\infty}p^{(2\mu_{i_j}-1)B_j(\mathbf{I}_1)}p^{-2(2\mu_{i_j}-1)A_j(\mathbf{I}_1)s}\right)$$

$$\times\prod_{\substack{j=0\\ j\notin\{i_1,\ldots,i_d\}}}^{k-1}\left(\sum_{\mu_j=1}^{\infty}p^{2\mu_jB_k(\mathbf{I}_1)}p^{-4\mu_jA_k(\mathbf{I}_1)s}\right)$$

$$=p^{\sum_{j=1}^{d}2A_{i_j}(\mathbf{I}_1)s-B_{i_j}(\mathbf{I}_1)}\prod_{j=1}^{k}(\zeta_p(4A_j(\mathbf{I}_1)s-2B_j(\mathbf{I}_1))-1). \tag{3.64}$$

∎

To state the aforementioned theorem, we adopt the notation from Section 3.4. Based on Lemma 3.9, we use the abbreviation

$$\sum_{(\sigma,I,k)} = \sum_{\substack{\sigma\in S_n\\ \sigma^2=\mathrm{id}}} \sum_{I=I_0\cup\cdots\cup I_s} \sum_{k=0}^{s+1},$$

put

$$\kappa_p(\sigma, I, k) = 2^{-c_1(\sigma)}(1-p^{-1})^{c_1(\sigma)+c_2(\sigma)}p^{-c_2(\sigma)}p^{-\tau(\{I_i\})-t(\sigma,\{I_i\})} \times \frac{2^{c_1^{(k)}(\sigma)}2^{\sum_{l=0}^{k-1}c_{1,l}}(1-p^{-1})^{c_1^{(k)(\sigma)}}p^{-\sum_{l=k+1}^{s}n(l)}p^{\sum_{l=0}^{k-1}(c_{1,l}+2c_{2,l})}}{\prod_{l=k}^{s}(1-p^{-n(l)})},$$

and further introduce

$$\Lambda_p(\sigma,\nu,q) = \begin{cases} 0 & \text{if } L'/L \cong \mathbb{Z}/p\mathbb{Z},\\ (-1)^{\sum_{l\in e(\nu)_k}c_{1,l}}\left(\frac{-1}{p}\right)^{((m-2)/2)\sum_{l\in o(\nu)_k}c_{1,l}} \times \left(\frac{2}{p}\right)^{\sum_{l\in e(\nu)_k}c_{1,l}} p^{m\sum_{l=0}^{k-1}S_l^{(k)}(\nu)((1/2)c_{1,l}+c_{2,l})} & \text{if } L'/L\cong(\mathbb{Z}/p\mathbb{Z})^2, \end{cases} \tag{3.65}$$

$$\Delta_p(\sigma,q) = \sum_{0\leqslant i_1\leqslant\cdots\leqslant i_d\leqslant n}(-1)^{\sum_{l\in e(i_1,\ldots,i_d)_k}c_{1,l}} \times\left(\frac{-1}{p}\right)^{((m-2)/2)\sum_{l\in o(i_1,\ldots,i_d)_k}c_{1,l}}\left(\frac{2}{p}\right)^{\sum_{l\in e(i_1,\ldots,i_d)_k}c_{1,l}} \times p^{\sum_{j=1}^{d}2A_{i_j}(\mathbf{I}_1)s-B_{i_j}(\mathbf{I}_1)}, \tag{3.66}$$

if $L'/L\cong(\mathbb{Z}/p\mathbb{Z})^2$ and zero if $L'/L\cong\mathbb{Z}/p\mathbb{Z}$.

THEOREM 3.12. *Let $(L,(\cdot,\cdot))$ be an even lattice of even rank m such that the discriminant form L'/L is anisotropic. Moreover, let p be an odd prime dividing $|L'/L|$, $\mu\in(L'/L)^n$, and $\Phi_p^{(\mu)}$ as defined in (3.23). Then*

$$M_p(s)\Phi_p^{(\mu)}(1,s) = \frac{\gamma_p(V)^n}{|L_p'/L_p|^{n/2}} \times \sum_{(\sigma,I,k)}\kappa_p(\sigma,I,k)\sum_{\{\nu\}_k}p^{\sum_{j=0}^{k-1}\nu_j(n(k)-n(j))}\Lambda_p(\sigma,\nu,q)X^{v(\nu)}\Phi_p^{(\mu)}(1,-s) \tag{3.67}$$

for all $s\in\mathbb{C}$ apart from poles on both sides, where $L_p=L\otimes\mathbb{Z}_p$ and $X=p^{-(s-s_0)}$ with $s_0=(m/2)-\rho_n$.

Proof. Let $L_{p,r}=L_p\oplus\mathbb{Z}_p^{2r}$. Associate to the p-part of μ the corresponding element in $(L_p'/L_p)^n$ as indicated in Remark 3.2 (which we also call μ). From Remark 3.2 and [MM, VI], it also follows that $(L_p,(\cdot,\cdot))$ is non-degenerate and L_p'/L_p is anisotropic.

According to [Ku2, Lemma A.3 and Proposition A.4] and [KY, Section 4.1], we have for all integers $r \geqslant \rho_n$

$$\int_{\mathrm{Sym}_n(\mathbb{Q}_p)} \Phi_p^{(\mu)}(wn(b), s_0 + r)\, db$$

$$= \gamma_p(V)^n \int_{\mathrm{Sym}_n(\mathbb{Q}_p)} \int_{\mu + L_{p,r}^n} \psi_p\left(\frac{1}{2}\operatorname{tr}(bQ^r[x])\right) dx\, db, \tag{3.68}$$

where $Q^r[x]$ is defined by (3.13) based on the quadratic form q_r (see (3.40)). The integral on the right-hand side of (3.68) can be evaluated by means of Lemma 3.9. Separating in (3.45) all parts depending on r from the rest, we obtain the formula in (3.67) for $s = r$, $r > \rho_n$, using the fact that $\Phi_p^{(\mu)}(1, -s) = \delta_{\mu,0}$. Lemma 3.11 guarantees that the right-hand side of (3.67) can be meromorphically continued to the complex plane. More specifically, from the explicit representation as a sum of finite products of local zeta functions, it follows that the right-hand side is a meromorphic function in p^{-s} on the complex plane. By [KR, Section 4], the same holds for $M_p(s)$. Combining these facts, we obtain (3.67) by analytic continuation (see also [Li2, p. 950]). □

We now make some special choices for $g_0 = (g_\tau, g_f) \in G(\mathbb{A})$ in order to define a vector valued Eisenstein series. To specify g_τ, let $\tau = u + iv \in \mathbb{H}_n$. Associated to τ, we set

$$g_\tau = n(u)m(a) \in \mathrm{Sp}(n, \mathbb{R}) \tag{3.69}$$

with ${}^t aa = v$. For all the finite places, we put $g_f = (1, 1, \ldots,)$. Moreover, let L be a lattice as introduced in Section 3.1.

Definition 3.13. Let $l \in 2\mathbb{Z}$ satisfy the condition $2l - b^- + b^+ \equiv 0 \bmod 4$, $\mu \in (L'/L)^n$ and $\mathfrak{e}_0 \in \mathbb{C}[(L'/L)^n]$.

(1) For $g_0 = (g_\tau, g_f) \in G(\mathbb{A})$ as above and $\Phi_\mu^l(s)$ the standard section in Definition 3.3, we define an S_L-valued Eisenstein series by

$$E_L(\tau, s, l) = \det(\mathrm{Im}(\tau))^{-l/2} \sum_{\mu \in (L'/L)^n} E(g_0, s, \Phi_\mu^l)\varphi_\mu. \tag{3.70}$$

(2) Further, using the notation $\Gamma_\infty = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}(n, \mathbb{Z}) \mid c = 0\right\}$ and $J\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau\right) = c\tau + d$, we define by

$$E_{l,0}^n(\tau, s) = \det(\mathrm{Im}(\tau))^s \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{Sp}(n,\mathbb{Z})} |\det(J(\gamma, \tau))|^{-2s} \det(J(\gamma, \tau))^{-l} \rho_{L,n}^{-1}(\gamma)\mathfrak{e}_0 \tag{3.71}$$

a vector valued non-holomorphic Siegel Eisenstein series $E_{l,0}^n$ of weight $l \in 2\mathbb{Z}$ transforming with the "finite" Weil representation $\rho_{L,n}$. We denote by $E_{l,0}^{n*}(\tau, s)$ the Eisenstein series in (3.71) with respect to the dual Weil representation $\rho_{L,n}^*$ analogous to the one defined in [BK, Section 3].

The next lemma states that there is a close relation between both Eisenstein series in Definition 3.13.

Lemma 3.14. *Let $E_L(\tau, s, l)$ be the Eisenstein series in (3.70) and $E_{l,0}^n(\tau, s)$ be the vector valued Eisenstein series in (3.71). Then the relation*

$$E_L(\tau, s, l) = E_{l,0}^n(\tau, (s + \rho_n - l)/2) \tag{3.72}$$

holds.

Proof. The proof in [BY, Section 2.2], can be generalized to our situation. The computation for the archimedean part of Φ_μ^l is the same as in [Ku1, Section IV.2]. ∎

REMARK 3.15. Using the scaled lattice $L(-1)$ instead of L, identity (3.72) reads as follows:

$$E_{L(-1)}(\tau, s, l) = E_{l,0}^{n*}(\tau, (s+\rho_n - l)/2). \tag{3.73}$$

The next theorem provides an explicit functional equation for the vector valued Eisenstein series $E_L(\tau, s, l)$. To this end, we define for a prime p

$$\begin{aligned} a_{n,p}(s) &= L_p(s+\rho_n - n, \chi_{V,p}) \prod_{k=1}^{[n/2]} \zeta_p(2s-n+2k), \\ b_{n,p}(s) &= L_p(s+\rho_n, \chi_{V,p}) \prod_{k=1}^{[n/2]} \zeta_p(2s+n-2k+1), \\ D_{p,j}(s) &= \zeta_p(4A_j(\mathbf{I_l})(s-s_0) - 2B_j(\mathbf{I_l})) \\ &\qquad \text{(see Lemma 3.11 for the notation } A_i(\mathbf{I_l}), B_i(\mathbf{I_l})) \end{aligned} \tag{3.74}$$

and associate to the set $P = \{q \text{ prime} \mid q \nmid |L'/L|\}$

$$a_n^P(s) = \prod_{p \in P} a_{n,p}(s), \qquad b_n^P(s) = \prod_{p\in P} b_{n,p}(s). \tag{3.75}$$

We finally put

$$\xi(s, n, l, P) = \frac{(-1)^{l/2} 2^{(1-s)n} \pi^{n(n+1)/2}}{|L'/L|^{n/2}} \frac{\Gamma_n(s)}{\Gamma_n(\alpha)\Gamma_n(\beta)} \frac{a_n^P(s)}{b_n^P(s)}. \tag{3.76}$$

THEOREM 3.16. *Let $E_L(\tau, s, l)$ be the Eisenstein series in Definition 3.13. Then $E_L(\tau, s, l)$ can be continued meromorphically to the whole complex s-plane. If additionally L'/L is anisotropic and $|L'/L|$ is odd, it satisfies the functional equation*

$$E_L(\tau, s, l) = \xi(s, n, l, P) \prod_{p \mid |L'/L|} \sum_{(\sigma, I, k)} \kappa_p(\sigma, I, k) \Delta_p(\sigma, q) \prod_{j=1}^{k} (D_{p,j}(s) - 1) E_L(\tau, -s, l). \tag{3.77}$$

Proof. The proof follows the one of [BY, Proposition 2.5].

Clearly, since each scalar valued Eisenstein series $E(g_0, s, \Phi_\mu^l)$ can be meromorphically continued to the whole complex s-plane, the same holds for $E_L(\tau, s, l)$.

It suffices to prove that the functional equation holds for each $E(g_0, s, \Phi_\mu^l)$. By (3.51), we have

$$E(g_0, s, \Phi_\mu^l) = E(g_0, -s, M(s)\Phi_\mu^l).$$

If $p = \infty$, Theorem 3.10 gives

$$\begin{aligned} M_\infty(s)\Phi_\infty^l(\gamma g_\tau, s) &= i^{-ln} 2^{(1-s)n} \pi^{n(n+1)/2} \frac{\Gamma_n(s)}{\Gamma_n(\alpha)\Gamma_n(\beta)} \Phi_\infty^l(\gamma g_\tau, -s) \\ &= \gamma_\infty(V)^n (-1)^{l/2} 2^{(1-s)n} \pi^{n(n+1)/2} \frac{\Gamma_n(s)}{\Gamma_n(\alpha)\Gamma_n(\beta)} \Phi_\infty^l(\gamma g_\tau, -s), \end{aligned} \tag{3.78}$$

where the last equation comes from $2l - b^- + b^+ \equiv 0 \bmod 4$ and the fact that $l \in 2\mathbb{Z}$.

If p is a prime not dividing $|L'/L|$, $\chi_{V,p}$ is unramified and $\Phi_p^{(\mu)}$ is the spherical standard section by Lemma 3.4. By [KR, Lemma 4.1], we then have

$$M_p(s)\Phi_p^{(\mu)}(g,s) = \frac{a_{n,p}(s)}{b_{n,p}(s)}\Phi_p^{(\mu)}(g,-s) \tag{3.79}$$

for all $g \in \mathrm{Sp}(n,\mathbb{Q}_p)$.

Now we consider the case of an odd prime p dividing $|L'/L|$. The local form (3.21) of $\Phi_p^{(\mu)}$ and formulas (3.25), (3.26), and (3.27) yield

$$\begin{aligned} &\Phi_p^{(\mu)}(gn(b)) = \psi_p(\mathrm{tr}(bQ[\mu]))\Phi_p^{(\mu)}(g), \quad b \in \mathrm{Sym}_n(\mathbb{Z}_p) \\ &\Phi_p^{(\mu)}(gw) = \gamma_p(V)^n |L_p'/L_p|^{-n/2} \sum_{\nu \in (L_p'/L_p)^n} \psi_p(\mathrm{tr}(\nu,\mu))\Phi_p^{(\nu)}(g) \\ &\Phi_p^{(\mu)}(gm(a)) = \chi_{V,p}(\det a)|\det a|_p^{m/2}\Phi_p^{(\mu a^{-1})}(g), \quad a \in \mathrm{GL}_n(\mathbb{Z}_p) \end{aligned} \tag{3.80}$$

for any $g \in \mathrm{Sp}(n,\mathbb{Q}_p)$. The definition of the intertwining operator then immediately implies that $M_p(s)$ satisfies the same equations as (3.25), (3.26), and (3.27). So, by Lemma 3.1, we arrive at

$$M_p(s)\Phi_p^{(\mu)}(k,s) = C_p\Phi_p^{(\mu)}(k,-s) \tag{3.81}$$

for all $k \in \mathrm{Sp}(n,\mathbb{Z}_p)$, where C_p is given by the right-hand side of (3.67). Clearly, since $M_p(s)\Phi_p^{(\mu)} \in I_n(-s,\chi_{V,p})$, the same equation holds for all $g \in \mathrm{Sp}(n,\mathbb{Q}_p)$. Taking Theorem 3.12 and notation (3.66), (3.58), and (3.64) respectively into account, we finally get

$$M_p(s)\Phi_p^{(\mu)}(g,s) = \frac{\gamma_p(V)^n}{|L_p'/L_p|^{n/2}} \sum_{(\sigma,I,k)} \kappa_p(\sigma,I,k)\Delta_p(\sigma,q)\prod_{j=1}^{k}(D_{p,j}(s)-1)\Phi_p^{(\mu)}(g,-s) \tag{3.82}$$

for all $g \in \mathrm{Sp}(n,\mathbb{Q}_p)$.

Combining (3.78), (3.79), and (3.82) together with the identities

$$\prod_{p \mid |L'/L|} |L_p'/L_p| = |L'/L|, \qquad \gamma_\infty(V)^n \prod_{p<\infty}\gamma_p(V)^n = 1,$$

the claimed functional equation follows. □

With the help of identity (3.72), we can immediately deduce that $E_{l,0}^n(\tau,s)$ possesses the same analytic properties as $E_L(\tau,s,l)$. In particular, we can deduce a functional equation for the Eisenstein series $E_{l,0}^n(\tau,s)$.

Corollary 3.17. *Let $E_{l,0}^n(\tau,s)$ be the Eisenstein series as in Definition 3.13 with respect to $\rho_{L,n}$. Then $E_{l,0}^n(\tau,s)$ has a meromorphic continuation in s to the whole complex plane. If L'/L is anisotropic and $|L'/L|$ is odd, it satisfies the functional equation*

$$\begin{aligned} E_{l,0}^n\left(\tau, s-\frac{l}{2}\right) &= \xi(2s-\rho_n,n,l,P)\prod_{p\mid |L'/L|}\sum_{(\sigma,I,k)}\kappa_p(\sigma,I,k)\Delta_p(\sigma,q) \\ &\times \prod_{j=1}^{k}(D_{p,j}(2s-\rho_n)-1)E_{l,0}^n\left(\tau,\rho_n-s-\frac{l}{2}\right). \end{aligned} \tag{3.83}$$

Proof. The assertion results immediately from the functional equation of the adelic Eisenstein series $E_L(\tau, s, l)$ and relation (3.72). □

REMARK 3.18. Let $L(-1)$ be the lattice as defined at the end of Section 3.3. It is easily checked that the integrals in Lemma 3.8 do not change by passing from q to $-q$ if L'/L is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Hence, the same holds for the formula in Lemma 3.9, which in turn implies that $E_{L(-1)}(\tau, s, l)$ satisfies the same functional equation as $E_L(\tau, s, l)$. Taking additionally (3.73) into account, we can deduce that $E^{n*}_{l,0}(\tau, s)$ obeys the same functional equation as $E^n_{l,0}(\tau, s)$.

§4. Jacobi modular forms of higher degree and L-functions

In this section, we will translate the functional equation (3.83) to Jacobi Eisenstein series of higher degree. To this end, we generalize in Section 4.1 a result of Arakawa [Ar1, Proposition 3.1], which establishes a relation between the real analytic Jacobi Eisenstein series and a certain vector valued Eisenstein series, to the corresponding real analytic Eisenstein series of higher degree. In doing so, we are able to transfer the analytic properties of the Eisenstein series (3.71) to Jacobi Eisenstein series of higher degree, provided the order of the involved discriminant group is odd. In Section 4.2, we employ the doubling method of Garrett [Ga] and Böcherer [Bo], which Arakawa carried over to the framework of Jacobi forms [Ar2], to obtain a functional equation for the standard zeta function $Z_n(s, f)$ of a Jacobi eigenform. To this end, we just need to apply the established functional equation of the Jacobi Eisenstein series to the identity in [Ar2, Theorem 2.9]. In Section 4.3, a functional equation and the meromorphic continuation to the complex s-plane of the standard L-function $L(s, f)$ are proved. Bouganis and Marzec proved under quite general assumptions a close relation between $Z_n(s, f)$ and $L(s, f)$; see [BM, Theorem 7.1]. In this subsection, we employ this relation and Corollary 4.6 in order to prove the desired analytic properties of $L(s, f)$. Note that these results are not new. Murase proved them in his papers [Mu1, Mu2], however in a quite different way. For $n = 1$, Sugano studied the standard L-function of a Jacobi form in [Su] under a much weaker assumption than [Mu1] and [Mu2].

4.1 Jacobi Eisenstein series

In order to define a Jacobi Eisenstein series, we need to introduce some notation and some basic facts about Jacobi forms, which we take from [Zi, Section 1] and [Ar2, Section 1].

The Jacobi group $G_{n,m}$ is the semidirect product of the Heisenberg group

$$H_{n,m} = \{[(\lambda, \mu), \kappa] \mid \lambda, \mu \in M_{m,n}, \kappa \in M_{m,m} \text{ where } (\kappa + \mu\lambda^t) \in \mathrm{Sym}_m\}$$

and $\mathrm{Sp}(n)$, that is,

$$G_{n,m} = H_{n,m} \rtimes \mathrm{Sp}(n).$$

Note that the group law on $H_{n,m}$ is given by

$$[(\lambda, \mu), \kappa] \cdot [(\lambda', \mu'), \kappa'] = [(\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + \lambda\mu'^t - \mu\lambda'^t]$$

and $\mathrm{Sp}(n)$ acts on the Heisenberg group by multiplication from the left

$$[(\lambda, \mu), \kappa] \circ M = [(\lambda, \mu)M, \kappa].$$

The product of two elements $([(\lambda, \mu), \kappa], M), ([(\lambda', \nu'), \kappa'], M') \in G_{n,m}$ reads accordingly as

$$([(\lambda, \mu), \kappa], M) \cdot ([(\lambda', \nu'), \kappa'], M') = ([(\lambda, \mu)M', \kappa] \cdot [(\lambda', \mu'), \kappa'], MM').$$

Note that $G_{n,m}$ can be interpreted as a subgroup of $\mathrm{Sp}(n+m)$ using suitable embeddings. Details can be found in [Zi] and [Ar2].

In accordance with (3.71), we set for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}(n, \mathbb{R})$ and $\tau \in \mathbb{H}_n$

$$M\langle\tau\rangle = (a\tau + b)(c\tau + d)^{-1} \qquad \text{and} \qquad J(M, \tau) = c\tau + d.$$

In terms of this notation, we can define an action of the real Jacobi group $G_{n,m}(\mathbb{R})$ on

$$\mathcal{H}_{n,m} = \mathbb{H}_n \times M_{m,n}(\mathbb{C})$$

by

$$g(\tau, z) := (M\langle\tau\rangle, zJ(M, \tau)^{-1} + \lambda M\langle\tau\rangle + \mu), \tag{4.1}$$

where $g = ([(\lambda, \mu), \kappa], M) \in G_{n,m}(\mathbb{R})$ and $(\tau, z) \in \mathcal{H}_{n,m}$. Let $S = (s_{ij})_{i,j} \in \mathrm{Sym}_m(\mathbb{Q})$ be a *positive definite half integral* matrix, which means that $2s_{ij}, s_{ii} \in \mathbb{Z}$. Attached to S we have a symmetric non-degenerate bilinear form on the lattice $\mathbb{Z}^m$:

$$(\lambda, \mu) = 2S(\lambda, \mu) \tag{4.2}$$

with the corresponding quadratic form

$$q(\lambda) = S[\lambda]$$

for $\lambda, \mu \in \mathbb{Z}^m$; see also (2.1).

Associated to S and a positive integer l, we specify a factor of automorphy

$$\begin{aligned} &J_{S,l}(g, (\tau, z)) \\ &= \det(J(M, \tau))^l e(-\operatorname{tr}(S\kappa) - \operatorname{tr}(S[\lambda]M\langle\tau\rangle + 2S(\lambda, zJ(M, \tau)^{-1}) - S[z]J(M, \tau)^{-1}c)) \end{aligned} \tag{4.3}$$

and a slash operator in terms of $J_{S,l}$ for a function $f : \mathcal{H}_{n,m} \to \mathbb{C}$ and $g \in G_{m,n}(\mathbb{R})$ by

$$(f\,|_{l,S}\, g)(\tau, z) = J_{S,l}(g, (\tau, z))^{-1} f(g(\tau, z)).$$

For later purposes, we introduce the concept of a Jacobi form of higher degree. Let

$$H_{n,m}(\mathbb{Z}) = \{[(\lambda, \mu), \kappa] \mid \lambda, \mu \in M_{m,n}(\mathbb{Z}), \kappa \in M_{m,m}(\mathbb{Z}) \text{ where } (\kappa + \mu\lambda^t) \in \mathrm{Sym}_m(\mathbb{Z})\}$$

and

$$\Gamma_{n,m} = H_{n,m}(\mathbb{Z}) \rtimes \mathrm{Sp}(n, \mathbb{Z}).$$

We call a holomorphic function $f : \mathcal{H}_{n,m} \to \mathbb{C}$ a *Jacobi form* of *weight* l and *index* S if f fulfills the following conditions:

(i) $(f\,|_{l,S}\,\gamma)(\tau, z) = f(\tau, z)$ for all $\gamma \in \Gamma_{n,m}$ and $(\tau, z) \in \mathcal{H}_{n,m}$.

(ii) $f(\tau, z)$ has a Fourier–Jacobi expansion of the form

$$f(\tau, z) = \sum_{\substack{N \in \mathrm{Sym}_n^*(\mathbb{Z}), r \in M_{m,n}(\mathbb{Z}) \\ N - (1/4) r^t S^{-1} r \geqslant 0}} c(N, r) e(\operatorname{tr}(N\tau + r^t z)), \tag{4.4}$$

where $\mathrm{Sym}_n^*(\mathbb{Z})$ is the set of all half integral matrices in $\mathrm{Sym}_n(\mathbb{R})$.

The condition on the Fourier expansion of f is only necessary in the case of $n = 1$. We denote by $J_{l,S}(\Gamma_{n,m})$ the space of all Jacobi forms of weight l and index S. As usual, $f \in J_{l,S}(\Gamma_{n,m})$ is a *cusp form* if it possesses a Fourier expansion where the condition $N - (1/4)r^t S^{-1} r \geqslant 0$ can be replaced by $N - (1/4)r^t S^{-1} r > 0$. By $J^{\mathrm{cusp}}_{l,S}(\Gamma_{m,n})$ we mean the subspace of all cusp forms.

Next, denote by $\Gamma^{\infty}_{n,m}$ the subgroup of $\Gamma_{n,m}$ given by

$$\Gamma^{\infty}_{n,m} = \left\{ \left([(\lambda, \mu), \kappa], \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \in \Gamma_{n,m} \mid c = 0, \lambda = 0 \right\}.$$

For an even integer $l \in \mathbb{N}$ and $(\tau, z) \in \mathcal{H}_{n,m}$, we define the real analytic Jacobi Eisenstein series

$$E^{(n)}_{l,S}(\tau, z; s) = \det(\mathrm{Im}(\tau))^s \sum_{\gamma \in \Gamma^{\infty}_{n,m} \backslash \Gamma_{n,m}} J_{S,l}(\gamma, (\tau, z))^{-1} |\det(J(M, \tau))|^{-2s}, \tag{4.5}$$

where $\gamma = ([(\lambda, \mu), \kappa], M)$. It can be proved that $E^{(n)}_{l,S}$ converges absolutely for $\mathrm{Re}(2s + l) > n + m + 1$; see [Zi, Theorem 2.1]. A complete set of coset representatives of $\Gamma^{\infty}_{n,m} \backslash \Gamma_{n,m}$ consists of

$$\left\{ \left([(\lambda a, \lambda b), 0], \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\infty} \backslash \mathrm{Sp}(n, \mathbb{Z}) \text{ and } \lambda \in M_{m,n}(\mathbb{Z}) \right\}; \tag{4.6}$$

see [Zi, Section 2].

In order to state the above-mentioned relation between both Eisenstein series, we need to introduce some theta series. For $\mu \in D_m(S) = ((2S)^{-1}\mathbb{Z}^m/\mathbb{Z}^m)^n$ $(\cong (2S)^{-1} M_{m,n}(\mathbb{Z})/M_{m,n}(\mathbb{Z}))$, the series

$$\theta_{S,\mu}(\tau, z) = \sum_{\lambda \in M_{m,n}(\mathbb{Z})} e(\mathrm{tr}(S[\lambda + \mu]\tau + 2S(\lambda + \mu, z))) \tag{4.7}$$

is analogous to the theta series as it is known for the classical Jacobi forms and which can be found for example in [Ar1, Section 1]. It converges normally on $\mathcal{H}_{n,m}$ and depends only on μ mod $M_{m,n}(\mathbb{Z})$. The transformation behavior of $\Theta_S(\tau, z) = \sum_{\mu \in D_m(s)} \theta_{S,\mu}(\tau, z)\mathfrak{e}_\mu$ under the action of the Jacobi group is well known. We have (see [Zi, Lemma 3.2] or [Ru, p. 168])

$$\Theta_S(M(\tau, z)) = e(\mathrm{tr}(S[z] J(M, \tau)^{-1} c)) \det(J(M, \tau))^{m/2} \rho_{L,n}(M) \Theta_S(\tau, z), \tag{4.8}$$

where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $M(\tau, z) = (M\langle\tau\rangle, zJ(M, \tau)^{-1})$ (apply (4.1) to $([(0, 0), 0], M)$) and $\rho_{L,n}$ is the Weil representation in (3.30) for the lattice $L = \mathbb{Z}^m$ equipped with the bilinear form (4.2).

Proposition 4.1. *Let $E^{(n)}_{l,S}(\tau, z, s)$ be the real analytic Jacobi Eisenstein series, $E^{n*}_{l,0}(\tau, s)$ the real analytic vector valued Eisenstein series (3.71) with respect to $\rho^*_{L,n}$, and $\Theta_S(\tau, s)$ the vector valued theta series as introduced before. Then the following relation between both Eisenstein series holds:*

$$E^{(n)}_{l,S}(\tau, z; s) = \langle E^{n*}_{l-(m/2),0}(\tau, s), \overline{\Theta_S(\tau, z)} \rangle, \tag{4.9}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product (3.31).

Proof. The proof is a straightforward generalization of the one of [Ar1, Proposition 3.1]. The Jacobi Eisenstein series $E_{l,S}^{(n)}(\tau, z; s)$ can be written in a more explicit way by making use of the set of coset representatives (4.6). As in [Zi, Section 2], we obtain for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$

$$\begin{aligned}
&E_{l,S}^{(n)}(\tau, z; s) \\
&\quad = \det(\operatorname{Im}(\tau))^s \sum_{\gamma \in \Gamma_\infty \backslash \operatorname{Sp}(n,\mathbb{Z})} |\det(J(\gamma,\tau))|^{-2s} \det(J(\gamma,\tau))^{-l} e(-\operatorname{tr}(S[z]J(\gamma,\tau)^{-1}c)) \\
&\qquad \times \sum_{\lambda \in M_{m,n}(\mathbb{Z})} e(\operatorname{tr}(S[\lambda]\gamma\langle\tau\rangle + 2S(\lambda, zJ(\gamma,\tau)^{-1}))) \\
&\quad = \det(\operatorname{Im}(\tau))^s \sum_{\gamma \in \Gamma_\infty \backslash \operatorname{Sp}(n,\mathbb{Z})} |\det(J(\gamma,\tau))|^{-2s} \det(J(\gamma,\tau))^{-l} \\
&\qquad \times e(-\operatorname{tr}(S[z]J(\gamma,\tau)^{-1}c))\theta_{S,0}(\gamma(\tau,z)) \\
&\quad = \det(\operatorname{Im}(\tau))^s \sum_{\gamma \in \Gamma_\infty \backslash \operatorname{Sp}(n,\mathbb{Z})} |\det(J(\gamma,\tau))|^{-2s} \det(J(\gamma,\tau))^{-l} \\
&\qquad \times e(-\operatorname{tr}(S[z]J(\gamma,\tau)^{-1}c))\langle \mathfrak{e}_0, \overline{\Theta_S(\gamma(\tau,z))}\rangle. \qquad (4.10)
\end{aligned}$$

Employing the transformation formula (4.8) and taking into account the identity $\overline{\rho_{L,n}} = \rho_{L,n}^*$ and that $\rho_{L,n}^*$ is unitary with respect to $\langle\cdot,\cdot\rangle$, we obtain for the right-hand side of (4.10)

$$\begin{aligned}
&\det(\operatorname{Im}(\tau))^s \sum_{\gamma \in \Gamma_\infty \backslash \operatorname{Sp}(n,\mathbb{Z})} |\det(J(\gamma,\tau))|^{-2s} \det(J(\gamma,\tau))^{-l+(m/2)} \langle \rho_{L,n}(\gamma)\mathfrak{e}_0, \overline{\Theta_S(\tau,z)}\rangle \\
&\quad = \langle E_{l-(m/2),0}^{n*}(\tau, s), \overline{\Theta_S(\tau,z)}\rangle.
\end{aligned}$$

∎

Relation (4.9) allows us to state the following analytic properties of $E_{l,S}^{(n)}(\tau, z; s)$ (which are partly already known) with respect to the variable s, provided that $\det(2S)$ is odd. We use the notation of Corollary 3.17.

THEOREM 4.2. *Let $m \in \mathbb{N}$ with $4 \mid m$, and let $S \in \operatorname{Sym}_m(\mathbb{Q})$ be positive definite and half integral. Then the real analytic Jacobi Eisenstein series can be meromorphically continued to the whole s-plane. If in addition, the quadratic module $(2S)^{-1}\mathbb{Z}^m/\mathbb{Z}^m$ is anisotropic and $\det(2S)$ odd, $E_{l,S}^{(n)}(\tau, z; s)$ satisfies the functional equation*

$$\begin{aligned}
E_{l,S}^{(n)}\left(\tau, z, s - \frac{l}{2} + \frac{m}{4}\right) &= \xi\left(2s - \rho_n, n, l - \frac{m}{2}, P\right) \prod_{p \mid |L'/L|} \sum_{(\sigma,I,k)} \kappa_p(\sigma, I, k)\Delta_p(\sigma, q) \\
&\quad \times \prod_{j=1}^{k} (D_{p,j}(2s - \rho_n) - 1) E_{l,S}^{(n)}\left(\tau, z, \rho_n - s - \frac{l}{2} + \frac{m}{4}\right). \qquad (4.11)
\end{aligned}$$

Proof. The assertion follows immediately from Corollary 3.17 and Proposition 4.1. ∎

4.2 Analytic properties of zeta functions of common Jacobi eigenforms

In this subsection, we prove analytic properties of a zeta function $Z_n(s, f)$ associated to a Hecke Jacobi eigenform f. To this end, we briefly recall the necessary background following [Ar2, Section 2] and [Bo, Section 3]. See also the references in [Ar2] for further details.

Let $\alpha \in \mathrm{Sp}(n, \mathbb{Q})$. The double coset $\Gamma_{m,n}\alpha\Gamma_{m,n}$ can be decomposed into a disjoint finite union of left cosets

$$\Gamma_{n,m}\alpha\Gamma_{n,m} = \bigcup_{i=1}^{r} \Gamma_{n,m}\beta_i.$$

It can be shown that by means of this decomposition, an action of double cosets $\Gamma_{m,n}\alpha\Gamma_{m,n}$ on $J_{l,S}(\Gamma_{n,m})$ is given by

$$(\Gamma_{n,m}\alpha\Gamma_{n,m}, f) \mapsto T(\alpha)f = f|\Gamma_{n,m}\alpha\Gamma_{n,m} = \sum_{i=1}^{r} f\,|_{l,S}\,\beta_i, \tag{4.12}$$

which stabilizes cusp forms. We now define a zeta function associated to a Hecke eigenform.

Definition 4.3. Let $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be an elementary divisor matrix, $d_n(D) = \begin{pmatrix} D & 0 \\ 0 & (D^{-1})^t \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{Q})$, and $f \in J_{l,S}^{\mathrm{cusp}}(\Gamma_{n,m})$ be a common Hecke eigenform of all Hecke operators $T(d_n(D))$, that is,

$$f|\Gamma_{n,m}d_n(D)\Gamma_{n,m} = \lambda(f, D)f$$

for all D. Then we can assign to f the Dirichlet series

$$Z_n(s, f) = \sum_{D \in D_n(\mathbb{Z})} \lambda(f, D)\det(D)^{-s}, \tag{4.13}$$

where $D_n(\mathbb{Z})$ denotes the set of elementary divisor matrices of rank n.

It can be shown that $Z_n(s, f)$ converges absolutely for $\mathrm{Re}(s) > 2n + m + 1$ [Ar2, Chapter 2]. Later, we will show that it can be continued meromorphically to the whole s-plane.

We need two further fundamental concepts to state the next result, which gives a representation of $Z_n(s, f)$ as a Rankin–Selberg convolution integral.

In the theory of the classical Siegel modular forms, the notion of an Eisenstein series is extended to the Klingen Eisenstein series attached to a cusp form. This construction can be generalized to Jacobi forms of higher degree.

Definition 4.4. Let $l \in \mathbb{N}$ be even, and $r \in \mathbb{Z}$ with $0 \leqslant r \leqslant n$. For $(\tau, z) \in \mathcal{H}_{n,m}$, set

$$(\tau^*, z^*) = \left(\tau\left[\begin{pmatrix} 0_{n-r} \\ 1_r \end{pmatrix}\right], z\begin{pmatrix} 0_{n-r} \\ 1_r \end{pmatrix}\right).$$

Moreover, depending on $f \in J_{l,S}^{\mathrm{cusp}}(\Gamma_{r,m})$, we define the function $\tilde{f}(\tau, z) = f(\tau^*, z^*)$ on $\mathcal{H}_{n,m}$ and the Klingen Eisenstein series

$$E_{l,S}^{n,r}(f, (\tau, z); s) = \sum_{\gamma \in \Gamma_{n,m}^r \backslash \Gamma_{n,m}} (\tilde{f}\,|_{l,S}\,\gamma)(\tau, z)\left(\frac{\det(\mathrm{Im}(M\langle\tau\rangle))}{\det(\mathrm{Im}(M\langle\tau\rangle^*))}\right)^s, \tag{4.14}$$

where $\gamma = ([(\lambda, \mu), \kappa], M)$ for each $\gamma \in \Gamma_{n,m}$ and

$$\begin{aligned}\Gamma_{n,m}^r = \{&([((0, \lambda_1), \mu), \kappa], M) \mid \\ &M \in \Gamma_{r,n}, \lambda_1 \in M_{m,r}(\mathbb{Z}), \mu \in M_{m,n}(\mathbb{Z}), \kappa \in M_{m,m}(\mathbb{Z}) \text{ where } (\kappa + \mu(0, \lambda_1)^t) \in \mathrm{Sym}_m(\mathbb{Z})\}\end{aligned}$$

with

$$\Gamma_{r,n} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{Z}) \,\middle|\, c = \begin{pmatrix} 0 & 0 \\ 0 & c_4 \end{pmatrix}, d = \begin{pmatrix} d_1 & 0 \\ d_3 & d_4 \end{pmatrix} \text{ with } c_4, d_4 \in M_{r,r}(\mathbb{Z})\right\}.$$

Ziegler showed that $E_{l,S}^{n,r}$ is absolutely convergent for $l + 2\,\mathrm{Re}(s) > n + m + r + 1$; see [Zi, Theorem 2.5].

The definition of a Petersson scalar product on $J_{l,S}^{\mathrm{cusp}}$ can be found in [Ar2] and [Zi]. Let $w = (\tau, z) \in \mathcal{H}_{n,m}$. We decompose τ and z into their real and imaginary parts, $\tau = x + iy$, $z = u + iv$ with $x, y \in \mathrm{Sym}_n(\mathbb{R})$ and $u, v \in M_{m,n}(\mathbb{R})$, and define $dw = d(\tau, z) = \det(y)^{-(m+n+1)}\, dx\, dy\, du\, dv$. Here $dx\, dy = \prod dx_{ij} \prod dy_{ij}$, and du and dv are the standard Haar measures on $M_{m,n}(\mathbb{R})$. Further, let

$$C_{S,l}(\tau, z) = \det(y)^l \exp(-4\pi \operatorname{tr}(S[v]y^{-1})).$$

For $f, g \in J_{l,S}(\Gamma_{n,m})$, one of which is a cusp form, the Petersson scalar product of f and g is given by

$$(f, g) = \int_{\Gamma_{n,m}\backslash\mathcal{H}_{n,m}} f(w)\overline{g(w)} C_{S,l}(w)\, dw. \tag{4.15}$$

We now describe the zeta function $Z_n(s, f)$ as a Rankin–Selberg convolution of a Jacobi Eisenstein series and the Jacobi cusp form f. To this end, let

$$\mu(l, S, n, s) = (-1)^{nl/2} 2^{n(n+1)-2ns-nl} \det(2S)^{-n} I_n\left(s + l - \frac{m}{2} - n - 1\right)$$

with

$$I_n(s) = \pi^{n(n+1)/2} 2^{-n(n-1)/2} \frac{\Gamma_n\left(s + 1 + \frac{n-1}{2}\right)}{\Gamma_n(s + n + 1)}.$$

The following theorem is taken from [Ar2, Section 2].

Theorem 4.5. *Let $p, n \in \mathbb{N}$ with $p \geqslant n$, $l \in \mathbb{N}$ even with $l > p + n + m + 1$, and $f \in J_{l,S}^{\mathrm{cusp}}(\Gamma_{n,m})$. If $l + 2\operatorname{Re}(s) > p + n + m + 1$, then for any $(\tau, z) \in \mathcal{H}_{p,m}$,*

$$\begin{aligned} &\int_{\Gamma_{n,m}\backslash\mathcal{H}_{n,m}} f(\zeta, w)\overline{E_{l,S}^{(p+n)}\left(\begin{pmatrix} -\overline{\tau} & 0 \\ 0 & \zeta \end{pmatrix}, (\overline{z}, w), \overline{s}\right)} C_{S,l}(\zeta, w) d(\zeta, w) \\ &\quad = \mu(l, S, n, s) \sum_{D \in D_n(\mathbb{Z})} (\det(D))^{-2s-l} E_{l,S}^{p,n}(f \mid \Gamma_{n,m} d_n(D) \Gamma_{n,m}, (\tau, z); s), \end{aligned} \tag{4.16}$$

where the integral on the left-hand side and the series on the right-hand side are absolutely convergent. Moreover, if f is a common eigenform, then the right-hand side of (4.16) coincides with

$$\mu(l, S, n, s) Z_n(2s + l, f) E_{l,S}^{p,n}(f, (\tau, z); s).$$

In particular, if $p = n$, this quantity is equal to

$$\mu(l, S, n, s) Z_n(2s + l, f) f(\tau, z). \tag{4.17}$$

We obtain as a corollary from Theorem 4.5 and identity (4.11) a functional equation for $Z_n(s, f)$.

Corollary 4.6. *Let $l > 2n + m + 1$ be an even positive integer, $m \in \mathbb{N}$ with $4|m$, and $f \in J_{l,S}^{\mathrm{cusp}}(\Gamma_{n,m})$ be a common eigenform of all Hecke operators $T(d_n(D))$. Then the Dirichlet series $Z_n(s, f)$ can be continued meromorphically to the whole s-plane. If $(2S)^{-1}\mathbb{Z}^m/\mathbb{Z}^m$ is anisotropic and* $\det(2S)$ *is odd, then*

$$\mathscr{Z}_n(s, f) = \mu(l, S, n, s) Z_n(2s + l, f)$$

satisfies the following functional equation,

$$\mathscr{Z}_n\left(s-\frac{l}{2}+\frac{m}{4},f\right)=\xi\left(2s-\rho_{2n},2n,l-\frac{m}{2},P\right)\prod_{p\mid\det(2S)}\sum_{(\sigma,I,k)}\kappa_p(\sigma,I,k)\Delta_p(\sigma,q)$$

$$\times\prod_{j=1}^{k}(D_{p,j}(2s-\rho_{2n})-1)\mathscr{Z}_n\left(\rho_{2n}-s-\frac{l}{2}+\frac{m}{4},f\right), \tag{4.18}$$

where the quantities $\xi(2s-\rho_{2n},2n,l-(m/2),P)$, $\kappa_p(\sigma,I,k)$, $\Delta_p(\sigma,q)$, *and* $D_{p,j}(2s-\rho_{2n})$ *are taken from Corollary 3.17.*

Proof. Note that $\overline{E^{2n*}_{l-(m/2),0}\left(\left(\begin{smallmatrix}-\overline{\tau}&0\\0&\zeta\end{smallmatrix}\right),\overline{s}\right)}=E^{2n}_{l-(m/2),0}\left(\left(\begin{smallmatrix}-\overline{\tau}&0\\0&\zeta\end{smallmatrix}\right),s\right)$ and therefore

$$\overline{E^{(2n)}_{l,S}\left(\begin{pmatrix}-\overline{\tau}&0\\0&\zeta\end{pmatrix},(\overline{z},w),\overline{s}\right)}=\left\langle E^{2n}_{l-(m/2),0}\left(\begin{pmatrix}-\overline{\tau}&0\\0&\zeta\end{pmatrix},s\right),\Theta_S\left(\begin{pmatrix}-\overline{\tau}&0\\0&\zeta\end{pmatrix},(\overline{z},w)\right)\right\rangle. \tag{4.19}$$

If we replace $\overline{E^{(2n)}_{l,S}\left(\left(\begin{smallmatrix}-\overline{\tau}&0\\0&\zeta\end{smallmatrix}\right),(\overline{z},w),\overline{s}\right)}$ in (4.16) by the right-hand side of (4.19), we find that $Z_n(s,f)$ is a meromorphic function in s whenever $l+2\operatorname{Re}(s)>2n+m+1$. Moreover, since $E^{2n}_{l-(m/2),0}$ can be continued to the whole s-plane, $Z_n(s,f)$ shares the same analytic property. The functional equation results from Corollary 3.17 in the same way. ∎

4.3 Analytic properties of the standard L-function

Let $f\in J^{\text{cusp}}_{l,S}$ be a common eigenform of all operators $T(d_n(\mathbb{Z}))$. We now briefly recall the theory necessary to define the standard L-function $L(s,f)$. More details can be found in [Ar2, BM], in particular in [Mu1] and [Mu2]. All of the following facts are based on an assumption imposed on the lattice $L=\mathbb{Z}_p^m$ with respect to the positive definite and half integral matrix $S\in\operatorname{Sym}_m(\mathbb{Q})$ for any prime p. It is denoted by M_p^+ in accordance with the notation in [Mu1] and [Mu2]:

(1) L is a $\mathbb{Z}_p$-maximal lattice with respect to $2S$.
(2) $L^*=L$, where $L^*=\{x\in L'=(2S)^{-1}L\mid S[x]\in(1/p)\mathbb{Z}_p\}$.

Note that if L is maximal, one can prove that $\operatorname{ord}_p(\det(2S))\in\{0,1,2\}$; see [Mu1, Lemma 4.1(i)]. This implies that $|L'/L|\in\{1,p,p^2\}$. Also, condition (2) guarantees that $S[x]\notin(1/p)\mathbb{Z}_p$ if $x\in L'\setminus L$. It follows that L'/L is anisotropic (provided it is non-trivial). Therefore, our assumption that L'/L is anisotropic is no further restriction compared to the condition M_p^+. As before, we consider only odd primes p.

Under the assumption M_p^+, the Hecke algebra $\mathcal{H}(G_{n,m}(\mathbb{Q}_p),G_{n,m}(\mathbb{Z}_p),\psi_{S,p})$ has a nice structure, as is proved in [Mu1, Section 4]. It is associated to a character $\psi_{S,p}:\operatorname{Sym}_m(\mathbb{Q}_p)\to\mathbb{C}^\times$, $\psi_{S,p}(\kappa)=\psi_p(\operatorname{tr}(S\kappa))$, where ψ_p is the p-part of a character $\psi:\mathbb{A}/\mathbb{Q}\to\mathbb{C}^\times$ with $\psi_\infty(x)=e(x)$ for $x\in\mathbb{R}$. $\mathcal{H}(G_{n,m}(\mathbb{Q}_p),G_{n,m}(\mathbb{Z}_p),\psi_{S,p})$ consists of functions $\phi:G_{n,m}(\mathbb{Q}_p)\to\mathbb{C}$, which fulfill the conditions:

(1)
$$\phi([(0,0),\kappa]kgk')=\psi_{S,p}(\kappa)\phi(g),\quad k,k'\in G_{n,m}(\mathbb{Z}_p),\ g\in G_{n,m}(\mathbb{Q}_p)\text{ and }\kappa\in\operatorname{Sym}_m(\mathbb{Q}_p),$$

(2) ϕ is compactly supported modulo

$$Z_{n,m}(\mathbb{Q}_p)=\{(0,0,\kappa)\mid\kappa\in\operatorname{Sym}_n(\mathbb{Q}_p)\}.$$

We will use the abbreviation $\mathcal{H}_{S,p}$ instead of $\mathcal{H}(G_{n,m}(\mathbb{Q}_p), G_{n,m}(\mathbb{Z}_p), \psi_{S,p})$. It can be shown that $\mathcal{H}_{S,p}$ forms a $\mathbb{C}$-algebra with respect to the convolution product

$$(\phi_1 * \phi_2)(g) = \int_{Z_{n,m}(\mathbb{Q}_p)\backslash G_{n,m}(\mathbb{Q}_p)} \phi_1(gx^{-1})\phi_2(x)\, dx, \quad \phi_1, \phi_2 \in \mathcal{H}_{S,p}, \tag{4.20}$$

where dx is the Haar measure on $Z_{n,m}(\mathbb{Q}_p)\backslash G_{n,m}(\mathbb{Q}_p)$ normalized by $\int_{Z_{n,m}(\mathbb{Q}_p)\backslash Z_{n,m}(\mathbb{Q}_p)G_{n,m}(\mathbb{Z}_p)} dx = 1$.

In order to define the standard L-function of a Hecke eigenform, we need to introduce zonal spherical functions ω_χ, where χ is an unramified character of

$$T(\mathbb{Q}_p) = \{d_m(\mathrm{diag}(d_1, \ldots, d_m)) \mid d_i \in \mathbb{Q}_p^\times\},$$

that is, it is an element of

$$X(T(\mathbb{Q}_p)) = \{\chi \in \mathrm{Hom}(T(\mathbb{Q}_p), \mathbb{C}^\times) \mid \chi \text{ is trivial on } T(\mathbb{Z}_p)\}.$$

Now, $\omega_\chi : G_{n,m}(\mathbb{Q}_p) \to \mathbb{C}$ is for $\chi = (\chi_1, \ldots, \chi_n) \in X(T(\mathbb{Q}_p))$ defined by

$$\omega_\chi(g) = \int_{G_{n,m}(\mathbb{Z}_p)} \phi_\chi(kg)\, dk.$$

Here ϕ_χ is another function on $G_{n,m}(\mathbb{Q}_p)$ given by

$$\begin{aligned}\phi_\chi &\left(\left([(0,\mu),\kappa], \begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix} d_n(\mathrm{diag}(d_1, \ldots, d_n))\right)([(\lambda,0),0],k)\right) \\ &= \psi_{S,p}(\kappa) \prod_{i=1}^n \chi_i(d_i)|d_i|_p^{2n+m+2-2i} \times \begin{cases} 1 & \text{if } \lambda \in M_{m,n}(\mathbb{Z}_p), \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Here $\lambda, \mu \in M_{m,n}(\mathbb{Q}_p)$, $\kappa \in \mathrm{Sym}_m(\mathbb{Q}_p)$, $x \in \mathrm{Sym}_n(\mathbb{Q}_p)$, $d_i \in \mathbb{Q}_p^\times$, and $k \in G_{n,m}(\mathbb{Z}_p)$. The following observation is crucial for the definition of $L(s,f)$: it is known that the map

$$\phi \mapsto \widehat{\omega_\chi}(\phi) = \int_{Z_{n,m}(\mathbb{Q}_p)\backslash G_{n,m}(\mathbb{Q}_p)} \omega_\chi(g)\phi(g^{-1})\, dg \tag{4.21}$$

gives rise to a $\mathbb{C}$-algebra homomorphism of $\mathcal{H}_{S,p}$ to $\mathbb{C}$. Moreover, every such homomorphism is of the form (4.21) for some uniquely determined $\chi \in X(T(\mathbb{Q}_p))$ up to some action on $X(T(\mathbb{Q}_p))$; see [Mu1].

We finally specify the standard L-function attached to a *cuspidal Jacobi form of weight l and index S*. To this end, we choose for each $(\tau, z) \in \mathcal{H}_{n,m}$ an element $g_\infty \in G_{n,m}(\mathbb{R})$ such that $g_\infty(i1_n, 0) = (\tau, z)$ and denote by $\mathscr{S}(l,S)$ the set of all bounded functions $f : G_{n,m}(\mathbb{A}) \to \mathbb{C}$ satisfying

(i)
$$f([(0,0),\kappa]\gamma g k_\infty k_f) = \det(J(k_\infty i1_n))^{-l}\psi_S(\kappa)f(g) \tag{4.22}$$

for $\kappa \in \mathrm{Sym}_m(\mathbb{A})$, $\gamma \in G_{n,m}(\mathbb{Q})$, $g \in G_{n,m}(\mathbb{A})$, $k_\infty \in K_\infty$, and $k_f \in K_f$, where K_∞ is given by (3.5) and K_f by (4.23);

(ii)
$$F_f(\tau, z) = f(g_\infty) J_{S,l}(g_\infty, (i1_n, 0))$$

is holomorphic on $\mathcal{H}_{n,m}$.

One can check that F_f does not depend on the choice of g_∞ and that for each $f \in \mathscr{S}(l, S)$, the associated function F_f is an element of $J_{k,S}^{\mathrm{cusp}}(\Gamma_{n,m})$. On the other hand, by means of the strong approximation theorem for $G_{n,m}$,

$$G_{n,m}(\mathbb{A}) = G_{n,m}(\mathbb{Q})G_{n,m}(\mathbb{R})K_f, \quad K_f = \prod_{p<\infty} G_{n,m}(\mathbb{Z}_p), \tag{4.23}$$

a cusp form $F \in J_{l,S}^{\mathrm{cusp}}(\Gamma_{n,m})$ can be related to $f \in \mathscr{S}(l, S)$ by

$$f(\gamma g_\infty k_f) = F(g_\infty\langle(i1_n, 0)\rangle)J_{l,S}(g_\infty, (i1_n, 0))^{-1}. \tag{4.24}$$

More precisely, the following statement holds ([Mu1], [Mu2], and [Ar2]).

Proposition 4.7. *The map $f \mapsto F_f$ is an isomorphism between the $\mathbb{C}$-vector spaces $\mathscr{S}(l, S)$ and $J_{l,S}^{\mathrm{cusp}}(\Gamma_{n,m})$.*

For each prime p, the Hecke algebra $\mathcal{H}_{S,p}$ acts on $\mathscr{S}(l, S)$ via convolution

$$(f * \phi)(g) = \int_{Z_{n,m}(\mathbb{Q}_p)\backslash G_{n,m}(\mathbb{Q}_p)} f(gx_p^{-1})\phi(x_p)dx_p, \quad f \in \mathscr{S}(l, S),\ \phi \in \mathcal{H}_{S,p}. \tag{4.25}$$

We call $f \in \mathscr{S}(l, S)$ a Hecke eigenform if for any prime p and any $\phi \in \mathcal{H}_{S,p}$, the equation

$$f * \phi = \lambda_{f,p}(\phi)f \quad \text{with } \lambda_{f,p}(\phi) \in \mathbb{C}$$

holds.

A Hecke eigenform f defines for each prime p a $\mathbb{C}$-algebra homomorphism $\lambda_{f,p} : \mathcal{H}_{S,p} \to \mathbb{C}$, $\phi \mapsto \lambda_{f,p}(\phi)$, and thereby determines a character $\chi_{f,p} = (\chi_{f,p}^{(1)}, \ldots, \chi_{f,p}^{(n)}) \in X(T(\mathbb{Q}_p))$ with

$$\lambda_{f,p}(\phi) = \widehat{\omega_{\chi_{f,p}}}(\phi), \quad \phi \in \mathcal{H}_{S,p}.$$

We are now able to define the standard L-function $L(s, f)$ attached to f by $L(s, f) = \prod_{p<\infty} L_p(s, f)$ with

$$L_p(s, f) = \prod_{i=1}^{n} [(1 - \chi_{f,p}^{(i)}(p)p^{-s})(1 - \chi_{f,p}^{(i)}(p)^{-1}p^{-s})]^{-1}. \tag{4.26}$$

There is a close relation between the action of $\mathcal{H}_{S,p}$ on $\mathscr{S}(l, S)$ and action (4.12) of a double coset $\Gamma_{n,m}d_n(D)\Gamma_{n,m}$ on $J_{l,S}^{\mathrm{cusp}}(\Gamma_{n,m})$ (see [Ar2]):

Let $\alpha = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n \geqslant 0$ and set $\pi_\alpha = \mathrm{diag}(p^{d_1}, \ldots, p^{d_n})$. Furthermore, we define an element $\phi_\alpha \in \mathcal{H}_{S,p}$ for $g = [(0, 0), \kappa]kgk' \in G_{n,m}(\mathbb{Q}_p)$ by

$$\phi_\alpha(g) = \begin{cases} \psi_{S,p}(\kappa) & \text{if } g \in Z_{n,m}(\mathbb{Q}_p)G_{n,m}(\mathbb{Z}_p)d_n(\pi_\alpha)G_{n,m}(\mathbb{Z}_p) \text{ and } g = [(0, 0), \kappa]kd_n(\pi_\alpha)k', \\ 0 & \text{if } g \notin Z_{n,m}(\mathbb{Q}_p)G_{n,m}(\mathbb{Z}_p)d_n(\pi_\alpha)G_{n,m}(\mathbb{Z}_p). \end{cases}$$

Arakawa points out that if $g \in Z_{n,m}(\mathbb{Q}_p)\Gamma_{n,m}d_n(\pi_\alpha)\Gamma_{n,m}$, the value $\phi_\alpha(g)$ does not depend on k, k', and κ. Moreover, it can be easily confirmed that ϕ_α indeed belongs to $\mathcal{H}_{S,p}$. In terms of ϕ_α, we can state the following proposition.

Proposition 4.8. [Ar2] *Let $F \in J_{k,S}^{\mathrm{cusp}}$ be a cusp form that corresponds to $f \in \mathscr{S}(l, S)$; see (4.24). Also, let $\alpha \in \mathbb{Z}^n$ be as above. Then for each $\phi_\alpha \in \mathcal{H}_{S,p}$, we have*

$$F_{f*\phi_\alpha} = F \mid \Gamma_{n,m}d_n(\pi_\alpha)\Gamma_{n,m}.$$

From Proposition 4.8, it follows immediately that $f \in \mathscr{S}(l, S)$ is a Hecke eigenform if and only $F_f \in J_{l,S}^{\mathrm{cusp}}$ is a common eigenform. Therefore, we define the standard L-function $L(s, F_f)$ of F_f to be $L(s, f)$.

We now want to transfer the analytic properties of the Dirichlet series $Z_n(s, f)$ to $L(s, f)$. Apart from Corollary 4.6, the result of [BM, Theorem 7.1] is crucial for our proof. It establishes a relation between $Z_n(s, f)$ and $L(s, f)$ for a more general situation than that considered in this paper. The factor $G_p(s)$ occurring in Theorem 7.1 is closely related to the Siegel series $b(S, s)$. In [Ka, Theorem 4.4], one can find an explicit formula for the Siegel series (note that $S \in \mathrm{Sym}_m(\mathbb{Q})$ is non-degenerate and half integral),

$$b(S, s) = \left(\zeta(s) \prod_{i=1}^{[m/2]} \zeta(2s-2i)\right)^{-1} L\left(s - \frac{m}{2}, \chi_S\right) \prod_{p \mid \det(2S)} g_p(S, p^{-s}), \tag{4.27}$$

where $L(s, \chi_S)$ is the L-function associated with the Hecke character corresponding to the extension $\mathbb{Q}(((-1)^{m/2}\det(S))^{1/2})/\mathbb{Q}$. The quantity $g_p(S, p^{-s})$ is a polynomial in p^{-s}, which is explicitly specified in [Ka, Theorem 4.3]. Now, the factor $G_p(s)$ is according to [BM] given by

$$G_p(s) = \frac{g_p(S, p^{-(s+(m/2)+n)})}{g_p(S, p^{-(s+(m/2))})}. \tag{4.28}$$

Tailored to our setting, Theorem 7.1 reads as follows.

THEOREM 4.9. [BM] *Let $0 \neq f \in J_{l,S}^{\mathrm{cusp}}(\Gamma_{n,m})$ be a Hecke eigenform of all Hecke operators $T(d_n(D))$. Assume further that the lattice $\mathbb{Z}_p^m$ with respect to the half integral matrix $S \in \mathrm{Sym}_m(\mathbb{Q}_p)$ satisfies the condition M_p^+ for every prime p. Then*

$$\mathfrak{L}(s) Z_n\left(s + \frac{m}{2} + n, f\right) = L(s, f), \tag{4.29}$$

where $\mathfrak{L}(s) = \prod_{p<\infty} \mathfrak{L}_p(s)$ with

$$\mathfrak{L}_p(s) = G_p(s) \prod_{i=1}^{n} \zeta_p(2s+2n-2i). \tag{4.30}$$

In particular, $L(s, f)$ is absolutely convergent for $\mathrm{Re}(s) > 2n+m+1$.

Combining Theorem 4.9 and Corollary 4.6 yields the desired analytic properties of $L(s, f)$ summarized in the next theorem.

THEOREM 4.10. *Let $l > 2n+m+1$ be an even positive integer, $m \in \mathbb{N}$ with $4 \mid m$, and $S \in \mathrm{Sym}_m(\mathbb{Q})$ be half integral for which $\mathbb{Z}_p^m$ satisfies the condition M_p^+ for every p. Moreover, let $f \in J_{l,S}^{\mathrm{cusp}}(\Gamma_{n,m})$ be a common Hecke eigenform of all Hecke operators $T(d_n(D))$. Then the standard L-function $L(s, f)$ can be continued meromorphically to the whole s-plane. If additionally, $\det(2S)$ is odd, then*

$$\Psi(s) = \mu\left(l, S, n, \frac{s+n}{2} - \frac{l}{2} + \frac{m}{4}\right) \mathfrak{L}^{-1}(s) L(s, f)$$

satisfies the functional equation

$$\Psi(s) = \xi\left(s - \frac{1}{2}, 2n, l - \frac{m}{2}, P\right) \prod_{p \mid \det(2S)} \sum_{(\sigma, I, k)} \kappa_p(\sigma, I, k) \Delta_p(\sigma, q)$$
$$\times \prod_{j=1}^{k} \left(D_{p,j}\left(s - \frac{1}{2}\right) - 1\right) \Psi(1-s). \tag{4.31}$$

Proof. Because of (4.29) we have

$$\mathscr{Z}_n(s,f) = \mu(l,S,n,s)\mathfrak{L}^{-1}\left(2s - \frac{m}{2} - n + l\right) L\left(2s - \frac{m}{2} - n + l, f\right).$$

Then the functional equation (4.18) for $\mathscr{Z}_n$ for $s/2 + n/2$ results in the functional equation

$$\begin{aligned} &\mu\left(l,S,n,\frac{s}{2}+\frac{n}{2}-\frac{l}{2}+\frac{m}{4}\right)\mathfrak{L}^{-1}(s)L(s,f) \\ &= \xi\left(s-\frac{1}{2}, 2n, l-\frac{m}{2}, P\right)\prod_{p\mid\det(2S)}\sum_{(\sigma,I,k)}\kappa_p(\sigma,I,k)\Delta_p(\sigma,q) \\ &\times \prod_{j=1}^{k}\left(D_{p,j}\left(s-\frac{1}{2}\right)-1\right)\mu\left(l,S,n,\frac{1-s}{2}+\frac{n}{2}-\frac{l}{2}+\frac{m}{4}\right) \\ &\times \mathfrak{L}^{-1}(1-s)L(1-s,f). \end{aligned} \tag{4.32}$$

∎

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