ON VECTOR VALUED AUTOMORPHIC FORMS AND THE STANDARD L-FUNCTION ATTACHED TO THEM

OLIVER STEIN

Abstract. We develop a theory of vector valued automorphic forms associated to the Weil representation ω_f and corresponding to vector valued modular forms transforming with the "finite" Weil representation ρ_L. For each prime p we determine the structure of a vector valued spherical Hecke algebra depending on ω_f, which acts on the space of automorphic forms. We finally define a standard L-function of such an automorphic form and prove that it can be continued meromorphically to the whole complex s-plane.

1. Introduction

An extremely important aspect of modular forms is their connection to L-functions, which in turn play a key role in number theory and beyond. There are several ways to associate an L-function to a modular form. Two examples of L-functions are relevant for the present paper. Both are associated to a common Hecke eigenform f. The first one is defined in terms of the eigenvalues $\lambda_D(f)$ of a family of Hecke operators $T(D)$, that is, $f \mid T(D) = \lambda_D(f)f$. It is given by

$$D(s,f) = \sum_D \lambda_D(f)\det(D)^{-s}. \tag{1.1}$$

The second one is constructed as an Euler product in terms of the Satake parameters $\alpha_{i,p}$ of f, where p is a prime. It can be written in the form $L(s,f) = \prod_{p<\infty} L_p(s,f)$ with

$$L_p(s,f) = R(\alpha_{i,p}, p^s),$$

where $R(\alpha_{i,p}, p^s)$ is a rational expression depending on the parameters $\alpha_{i,p}$ and the prime power p^s. One way to obtain $R(\alpha_{i,p}, p^s)$, is to express the formal local power series $\sum_D T(D)X^{r(D)}$ as a rational function in X as was done in [Bo2], [BM], [BoSP] or [Sh1].

Vector valued modular forms transforming with the Weil representation play an important role many recent papers. The weakly holomorphic forms of this type serve as input to a singular theta lift, which maps them to meromorphic modular forms on orthogonal groups whose zeroes and poles are supported on special divisors and which possess an infinite product expansion. This theta lift is the celebrated Borcherds lift ([Bo], [Br1]), which has many applications in geometry, algebra and in the theory of Lie algebras. For instance, it is an interesting and widely studied problem to classify reflective automorphic forms and thereby so-called reflective lattices and Kac-Moody algebras (see e. g. [Sch1] and [Wa]). We believe that the results of the present paper can contribute to these studies. Most of the classical theory of modular forms has been established for this type of modular forms over the past years. However, still not much is known about associated Dirichlet series. In [BS], a zeta function of the form (1.1) was introduced. In [St1] the analytic properties of a slightly more

1991 *Mathematics Subject Classification.* 11F27 11F25 11F41 11F66.

general zeta function were investigated. So far, to the best of my knowledge, there has been no standard L-function defined for this type of modular forms. The main objectives of the present paper are

i) to develop a theory of vector valued automorphic forms associated to the Weil representation ω_f corresponding to vector valued modular forms transforming according to the finite Weil representation,
ii) to determine the structure of a local vector valued spherical Hecke algebra depending on ω_f,
iii) to define such a standard L-function assigned to a vector valued Hecke eigenform f and to prove that it is meromorphic on the complex s-plane.

To achieve this, we mainly follow Murase ([Mu1], § 4), Bouganis and Marzec ([BM], Chapter 7) and Böcherer and Schulze-Pillot ([BoSP], I.§2) (which provide a thorough account to this topic). As in [Mu1], we first develop a theory of vector valued automorphic forms for the Weil representation and prove that the space of these forms is isomorphic to the space of vector valued cusp forms for the Weil representation. This may be of interest in its own right. We then define for each prime p a spherical Hecke algebra depending on the Weil representation and an action of this algebra on the space automorphic forms, which is compatible with the action of Hecke operators (as defined in [BS]) on the space of cusp forms. For a common automorphic eigenform F of all elements $T_{k,l}$ of the before mentioned spherical Hecke algebra we are then able to produce a result of the form

$$\sum_{(k,l)} \lambda_{F,p}(T_{k,l})p^{-(k+l)} = R(\alpha_{1,p}, \alpha_{2,p}, p^s),$$

where the Satake parameters are given by two unramified characters of the p-adic numbers $\mathbb{Q}_p$. As was suggested in [BM] and [BoSP], we use this identity to define a standard L-function of F. The last part of this paper is concerned with the proof of the meromorphic continuation of this L-function to the whole s-plane.

Note that the latter part may already be covered in a paper of Yamana ([Ya]), who proved analytic properties of a standard L-function associated to an automorphic representation of the metaplectic group $\mathrm{Mp}(2n)$ (see the remarks below regarding this issue). Yet, these results seem not to be immeadiately applicable to our setting and it is unclear how to recover the computations of the local Euler factors from them (which play an important role in applications - see the remarks below).

The paper at hand is intended as a first step towards a more comprehensive study of L-functions associated to vector valued automorphic forms for the Weil representation: Scheithauer (see [Sch] for example) and others investigated extensively a lifting from scalar valued modular forms for $\Gamma_0(N)$, N the level of the lattice L (see below for more details), to vector valued modular forms transforming with the Weil representation, which commutes with Hecke operators on both sides. On the other hand, it is well known that there is a well established theory of automorphic forms and automorphic representations of $\mathrm{GL}(2)$ connected to modular forms of $\Gamma_0(N)$ (see e. g. [Ge]). It would be interesting to compare a vector valued automorphic form obtained from a lifted scalar valued modular form with the corresponding scalar valued automorphic form. I expect that there is a relation between the associated standard L-functions, which in turn could lead to a relation of the standard L-function in this paper and the one related to some irreducible automorphic representation of $\mathrm{GL}(2)$. I hope to come back these questions in the near future. Also, this paper (and [St1]) can be used to prove more general results on the injectivity of the Kudla-Millson lift along the lines of [BF]. The corresponding proof relies on the computation of the local Euler factors of the introduced standard L-function. In turn, a generalization of a converse theorem for lattices of level p as stated in [Br2] can be proven. This is indeed achieved in [St2]. Although our results are restricted to anisotropic discriminant forms, based on them, we have established the most general converse theorem for the Borcherds lift (to the best of my knowlegde).

Let us describe the content of the paper in more detail. To this end, let $(L, (\cdot,\cdot))$ be an even lattice of even rank m and type (b^+, b^-) with (even) signature $\operatorname{sig}(L) = b^+ - b^-$ and level N. Associated to the bilinear form $(\cdot,\cdot)$ there is a quadratic form q. The modulo 1 reduction of $(\cdot,\cdot)$ and q defines a bilinear form and quadratic form, respectively, on the discriminant form $D = L'/L$. Here L' is the dual lattice of L. The Weil representation ρ_L is a representation of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ on the group ring $\mathbb{C}[D]$. As usual, denote with $\mathbb{Z}_p$ the ring of p-adic integers. As will be explained later in the paper, ρ_L is isomorphic to a finite dimensional subrepresentation of the Weil representation $\omega_f = \bigotimes_{p<\infty} \omega_p$ (originally defined by Weil [Wei]) of $\mathrm{SL}_2(\widehat{\mathbb{Z}})$ on a space S_L (isomorphic to $\mathbb{C}[D]$). Here $\widehat{\mathbb{Z}}$ is defined by $\prod_{p<\infty} \mathbb{Z}_p$. We have the relation

$$\rho_L(\gamma) = \omega_f(\gamma_f)$$

for all $\gamma \in \Gamma$, where γ_f is the projection of γ into $\mathrm{SL}_2(\widehat{\mathbb{Z}})$. For the details see Chapter 4.

For $\kappa \in \mathbb{Z}$, a vector valued modular form of weight κ and type ρ_L is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$, which satisfies

$$f(\gamma\tau) = (c\tau + d)^\kappa \rho_L(\gamma) f(\tau)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all τ in the complex upper half plane $\mathbb{H}$, and is holomorphic at the cusp ∞. We denote the space of all such functions with $M_\kappa(\rho_L)$ and write $S_\kappa(\rho_L)$ for the subspace of cuspforms. Now let $\mathbb{A}$ be the ring of adeles, $\mathcal{G}(\mathbb{Q})$ a subgroup of $\mathrm{GL}_2(\mathbb{Q})^+$ and

$$\mathcal{G}(\mathbb{A}) = {\prod_{p\le\infty}}' \mathcal{Q}_p = \left\{ (g_p) \in \prod_{p\le\infty} \mathcal{Q}_p \mid g_p \in \mathcal{K}_p \text{ for almost all primes } p \right\},$$

where $\mathcal{Q}_p$ and $\mathcal{K}_p$ is a subgroup of $GL_2(\mathbb{Q}_p)$ and $\mathrm{GL}_2(\mathbb{Z}_p)$, respectively. We assign to f a function $F_f : \mathcal{G}(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{A}) \to \mathbb{C}$ by means of strong approximation for the group $\mathcal{G}(\mathbb{A})$. For $g = g_{\mathbb{Q}}(g_\infty \times k)$ we put

$$F_f(g) = \omega_f^{-1}(k) j(g_\infty, i)^{-\kappa} f(g_\infty i).$$

Here $g_{\mathbb{Q}} \in \mathcal{G}(\mathbb{Q})$, $g_\infty \in \mathcal{Q}_\infty < \mathrm{GL}_2^+(\mathbb{R})$ and $k \in \mathcal{K} = \prod_{p<\infty} \mathcal{K}_p$. In Proposition 6.3 and Lemma 6.4 we will show that F is a cuspidal vector valued automorphic form of type ω_f, which can be seen as a vector valued analogue of a scalar valued cuspidal automorphic form. Moreover, Theorem 6.5 states that space of these functions, satisfying further properties, is isomorphic to $S_\kappa(\rho_L)$. We denote this space with $A_\kappa(\omega_f)$. The inverse map can also be explicitly given: For $F \in A_\kappa(\omega_f)$ it can be proven that f_F, specified by

$$\tau \mapsto f_F(\tau) = j(g_\tau, i)^\kappa F(g_\tau \times 1_f)$$

with $g_\tau = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$ and $\tau = g_\tau i = x + iy \in \mathbb{H}$, is indeed an element of $S_\kappa(\rho_L)$. Note that the definition of F_f has already occurred in the work of Werner ([We]). The function f_F can be found in Kudla's paper [Ku]. However, as far as I know, a theory of vector valued automorphic forms has not yet appeared in the literature.

It is well known that for each prime p the spherical Hecke algebra $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p)//\mathrm{GL}_2(\mathbb{Z}_p))$ of locally constant, compactly supported and complex-valued functions, which additionally satisfy

$$f(k_1 g k_2) = f(g)$$

for all $k_1, k_2 \in \mathrm{GL}_2(\mathbb{Z}_p)$ and all $g \in \mathrm{GL}_2(\mathbb{Q}_p)$, acts on the space of automorphic forms. This action is compatible with action of Hecke operators on the space of cusp forms (see for example [BP], [Ge] or [KL]). Werner introduced in [We] an action on $A_\kappa(\omega_f)$, which is compatible with the action of the double coset $\Gamma \left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) \Gamma$ on $S_\kappa(\rho_L)$. In this paper, we extend Werner's result. We define for each prime p a spherical Hecke algebra $\mathcal{H}(\mathcal{Q}_p)//\mathcal{K}_p, \omega_p)$ of type ω_p as follows: Let $L_p = L \otimes \mathbb{Z}_p$ and S_{L_p} as above, but associated to the p-adic lattice L_p. Note that $S_L = \bigotimes_{p<\infty} S_{L_p}$ (see Chapter 4 for details). Then $\mathcal{H}(\mathcal{Q}_p)//\mathcal{K}_p, \omega_p)$ consists of all locally constant and compactly supported functions $f : \mathcal{Q}_p \to S_{L_p}$, which satisfy

$$f(k_1 g k_2) = \omega_p(k_1) \circ f(g) \circ \omega_p(k_2)$$

for all $k_1, k_2 \in \mathcal{K}_p$ and all $g \in \mathcal{Q}_p$. Hecke algebras of this type are well known and studied in the literature ([BK], [Ho], [He]). A "tool" to investigate the structure of Hecke algebras (for a pair of groups (G, K) with suitable properties), vector valued or not, is the Satake map (see [Sa] or [Ca]), whose image is a Hecke algebra easier to understand. Under the assumption that L_p'/L_p is *anisotropic*, we determine a set of generators of $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ and connect it to a scalar valued Hecke algebra by means of a variant of the classical Satake map, which we adopt from [He] and which is suitable in our situation. Subsequently, we define an action of $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ on $A_\kappa(\omega_f)$, which can be interpreted as vector-valued analogue of the versions in [BP] or [Mu1]. We show in Theorem 6.9 that this action is compatible with the action of Hecke operator on $S_\kappa(\rho_L)$.

Denote with $T_{k,l}$, $(k, l) \in \mathbb{Z}^2$ with $l \geq k \geq 0$ and $k + l \in 2\mathbb{Z}$, a set of generators of $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ and $F \in A_\kappa(\omega_f)$ a common eigenform of all $T_{k,l}$. Following Arakawa in [Ar], we transfer some results on spherical functions to the vector valued setting and use them to prove

$$\sum_{(k,l)\in\Lambda_+} \lambda_{F,p}(T_{k,l}) p^{-s(k+l)} = \frac{1}{C(L_p)} \frac{1 + \chi^{(1)}_{F,p}(p)\chi^{(2)}_{F,p}(p)p^{-2s+1}}{(1 - \chi^{(1)}_{F,p}(p^2)p^{-2s+1})(1 - \chi^{(2)}_{F,p}(p^2)p^{-2s+1})},$$

where $(\chi^{(1)}_{F,p}, \chi^{(2)}_{F,p})$ is a pair of unramified characters of $\mathbb{Q}_p$ and $C(L_p)$ is some constant. As indicated above, this identity gives rise to our definition of the local standard L-function $L_p(s, F)$.

The last part of the paper deals with the analytic properties of $L(s, F)$. Following [Ar] again, we find a direct relation between the standard L-function and a zeta function $\mathcal{Z}(s, f)$ similar to the one in (1.1). In [St1] a variant of the before mentioned zeta function was studied. It turns out that it is a meromorphic function and satisfies a functional equation. The property of being meromorphic carries over to $L(s, F)$, the functional equation does unfortunately not. The local L-function $L_p(s, F)$ possesses an integral representation of the form

$$\int_{\mathcal{Q}_p} \nu_{s+\frac{1}{2}}(g)\omega_\chi(g)dg, \tag{1.2}$$

where ν_s is defined in (7.14) and ω_χ is the zonal spherical function associated to the pair of characters $(\chi^{(1)}_{F,p}, \chi^{(2)}_{F,p})$. The analytic properties of integrals of this type (as a function of s) were subject of several papers, among them [Ta] and [An]. The main difference between (1.2) and these integrals is that the one considered in the cited papers are defined over a multiplicative group of an algebra, which is not the case for (1.2). As consequence, the tools used in [Ta] and [An] to prove a functional equation, are not at our disposal for this task. It seems (to the best of my knowledge) unclear to date how they can be replaced. It would be interesting to investigate this subject further and to provide the means to prove a functional equation in the present situation.

2. Notation

As usual, we let $e(z)$, $z \in \mathbb{C}$ be the abbreviation for $e^{2\pi i z}$. For any prime $p \in \mathbb{Z}$ by $\mathbb{Q}_p$ we mean the field of p-adic numbers and by $\mathbb{Z}_p$ its ring of p-adic integers; $|\cdot|_p$ is the p-adic absolute value and $\mathrm{ord}_p(\cdot)$ the p-adic valuation of $\mathbb{Q}_p$. We write $\mathbb{A}$ for the adele ring and for $\mathbb{A}^\times$ for the idele group. By $\mathbb{A}_f$ we mean the set of finite adeles. For any ring R, as usual, $M_{2,2}(R)$ and $\mathrm{GL}_2(R)$ are the set of 2×2 matrices, the subgroup of invertible matrices in $M_{2,2}(R)$. The following subgroups are important for this paper:

$$\begin{aligned}
\mathcal{G}(R) &= \{M \in GL_2(R) \mid \det(M) \in (R^\times)^2\},\\
D(R) &= \{\left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1}\end{smallmatrix}\right) \mid r \in R^\times\},\\
M(R) &= \{\left(\begin{smallmatrix} r_1 & 0 \\ 0 & r_2\end{smallmatrix}\right) \mid r_1, r_2 \in R^\times\},\\
N(R) &= \{\left(\begin{smallmatrix} 1 & r \\ 0 & 1\end{smallmatrix}\right) \mid r \in R\},\\
U(R) &= \{\left(\begin{smallmatrix} 1 & 0 \\ r & 1\end{smallmatrix}\right) \mid r \in R\} \text{ and}
\end{aligned}$$

$B(R) = M(R) \cdot N(R) = N(R) \cdot M(R)$. Throughout the paper we use the following abbreviations for certain elements of these groups

$$n_{-}(c) = \left(\begin{smallmatrix} 1 & 0 \\ c & 1\end{smallmatrix}\right), \quad n(b) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right), \quad m(s) = \left(\begin{smallmatrix} s & 0 \\ 0 & s^{-1}\end{smallmatrix}\right), \quad m(t_1, t_2) = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2\end{smallmatrix}\right) \text{ and } w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right).$$

For $R = \mathbb{Q}_p$ or $R = \mathbb{Z}_p$ the groups listed below will be frequently be utilized in this paper.

$$\begin{aligned}
\mathcal{Q}_p &= \{M \in \mathrm{GL}_2(\mathbb{Q}_p) \mid \det(M) \in (\mathbb{Q}_p^\times)^2\},\\
\mathcal{K}_p &= \{M \in \mathrm{GL}_2(\mathbb{Z}_p) \mid \det(M) \in (\mathbb{Z}_p^\times)^2),\\
\mathcal{M}_p &= \{M \in M(\mathbb{Q}_p) \mid \det(M) \in (\mathbb{Q}_p^\times)^2\},\\
\mathcal{D}_p &= \mathcal{M}_p \cap \mathcal{K}_p \text{ and}\\
\mathcal{N}_p &= \{\left(\begin{smallmatrix} r_1 & 0 \\ 0 & r_2\end{smallmatrix}\right) \in M(\mathbb{Z}_p) \mid r_1 \in (\mathbb{Z}_p^\times)^2, r_2 = 1\}.
\end{aligned} \tag{2.1}$$

Moreover, the subgroup

$$\mathcal{K}_0(p) = \left\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathcal{K}_p \mid c \equiv 0 \bmod p\right\} \tag{2.2}$$

of $\mathcal{K}_p$ will be relevant. Since $\mathcal{Q}_p$ is locally compact (see Lemma 5.3), we may fix a Haar measure on $\mathcal{Q}_p$ such that $\int_{G \cap \mathcal{K}_p} dg = 1$ for any of the groups $G = \mathcal{Q}_p, \mathcal{M}_p$ and $N(\mathbb{Q}_p)$. We denote with $\mu(K)$ the measure of any subgroup K of $\mathcal{K}_p$.

For $x_p \in \mathcal{Q}_p$ let $\iota_p(x_p) = (\alpha_q)_{q \leq \infty} \in \mathrm{GL}_2(\mathbb{A})$ with $\alpha_q = 1_q$ for $q \neq p$ and $\alpha_p = x_p$.

For an arbitrary set A, by $\mathbb{1}_A$ we mean the characteristic function of A. Moreover, we will make frequently use of the following subsets of $\mathbb{Z}^2$:

$$\Lambda = \{(k,l) \in \mathbb{Z}^2 \mid k,l \geq 0 \text{ and } k+l \in 2\mathbb{Z}\} \text{ and}$$
$$\Lambda_+ = \{(k,l) \in \Lambda \mid k \leq l\}.$$

Finally, as usual, we write $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$ for the complex upper half plane and $\left(\frac{\cdot}{d}\right)$ for the Legendre symbol.

3. Hecke operators $T(m(p^k, p^l))$ and standard zeta-functions

In this section we briefly summarize some facts on lattices, discriminant forms and the "finite" Weil representation. We also recall the definition of vector valued modular forms for the Weil representation and some related theory relevant for the present paper. Subsequently, we explain how to extend the definition of the Hecke operator $T(p^{2l})^*$ in Definition 5.5 of [BS] to the action of double cosets of the form $\Gamma m(p^k, p^l)\Gamma$. In [BS] a standard zeta function associated to an eigenform of all Hecke operators $T(m(n^2, 1))$, $n \in \mathbb{N}$ is introduced. Here, we consider a variant of this zeta function and state some analytic properties based on the investigations in [St1].

Let L be a lattice of rank m equipped with a symmetric $\mathbb{Z}$-valued bilinear form $(\cdot,\cdot)$ such that the associated quadratic form

$$q(x) := \frac{1}{2}(x,x), \quad x \in L,$$

takes values in $\mathbb{Z}$. We assume that m is even, L is non-degenerate and denote its type by (b^+, b^-) and signature $b^+ - b^-$ by $\operatorname{sig}(L)$. Note that $\operatorname{sig}(L)$ is also even. We stick with these assumptions on L for the rest of this paper unless we state it otherwise. Further, let

$$L' := \{x \in V = L \otimes \mathbb{Q} \; : \; (x,y) \in \mathbb{Z} \quad \text{for all } \; y \in L\}$$

be the dual lattice of L. Since $L \subset L'$, the elementary divisor theorem implies that L'/L is a finite group. We denote this group by D. The modulo 1 reduction of both, the bilinear form $(\cdot,\cdot)$ and the associated quadratic form, defines a $\mathbb{Q}/\mathbb{Z}$-valued bilinear form $(\cdot,\cdot)$ with corresponding $\mathbb{Q}/\mathbb{Z}$-valued quadratic form on D. We call D combined with $(\cdot,\cdot)$ a discriminant form or a quadratic module. We call it anisotropic, if $q(\mu) = 0$ holds only for $\mu = 0$.

It is well known that any discriminant form can be decomposed into direct sum of quadratic modules of the following form (cf. [BEF])

$$\mathcal{A}^t_{p^k} = \left(\mathbb{Z}/p^k\mathbb{Z}, \; \frac{tx^2}{p^k}\right), \; p > 2, \quad \mathcal{A}^t_{2^k} = \left((\mathbb{Z}/2^k\mathbb{Z}, \; \frac{tx^2}{2^{k+1}}\right),$$
$$\mathcal{B}_{2^k} = \left(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}; \; \frac{x^2+2xy+y^2}{2^k}\right), \quad \mathcal{C}_{2^k} = \left(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}; \; \frac{xy}{2^k}\right).$$

The structure of anisotropic finite quadratic modules is well known: In particular, for an odd prime p each p-group D_p of a discriminant form D can be either written as $\mathcal{A}^t_p$ or as a direct sum $\mathcal{A}^t_p \oplus \mathcal{A}^1_p$. For further details we refer to [BEF].

The Weil representation ρ_L is a representation of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ on the group ring $\mathbb{C}[D]$. We denote the standard basis of $\mathbb{C}[D]$ by $\{\mathfrak{e}_\lambda\}_{\lambda \in D}$. The standard scalar product on $\mathbb{C}[D]$ is given by

$$\left\langle \sum_{\lambda\in D} a_\lambda \mathfrak{e}_\lambda, \sum_{\lambda\in D} b_\lambda \mathfrak{e}_\lambda \right\rangle = \sum_{\lambda\in D} a_\lambda \overline{b_\lambda}. \tag{3.1}$$

As Γ is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{3.2}$$

it is sufficient to define ρ_L by the action on these generators.

Definition 3.1. The representation ρ_L of Γ on $\mathbb{C}[D]$, defined by

$$\begin{aligned} \rho_L(T)\mathfrak{e}_\lambda &:= e(bq(\lambda))\mathfrak{e}_\lambda, \\ \rho_L(S)\mathfrak{e}_\lambda &:= \frac{e(-\operatorname{sig}(L)/8)}{|D|^{1/2}} \sum_{\mu\in D} e(-(\mu,\lambda))\mathfrak{e}_\mu, \end{aligned} \tag{3.3}$$

is called Weil representation.

Let $Z = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The action of Z is given by

$$\rho_L(Z)(\mathfrak{e}_\lambda) = e(-\operatorname{sig}(L)/4)\mathfrak{e}_{-\lambda}. \tag{3.4}$$

We denote by N the level of the lattice L. It is the smallest positive integer such that $Nq(\lambda) \in \mathbb{Z}$ for all $\lambda \in L'$. One can prove that the Weil representation ρ_L is trivial on $\Gamma(N)$, the principal congruence subgroup of level N. Therefore, ρ_L factors over the finite group

$$\Gamma/\Gamma(N) \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$

For the rest of this paper we suppose that N is *odd*.

The following Lemma describes the action of the Weil representation on matrices of the form

$$R_d = ST^dS^{-1}T^aST^d,$$

where $ad \equiv 1 \bmod N$. It es easily checked that $R_d \equiv \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \bmod N$.

Lemma 3.2. *For a, d as above we have*

$$\rho_L(R_d)\mathfrak{e}_\lambda = \frac{g_d(D)}{g(D)}\mathfrak{e}_\lambda. \tag{3.5}$$

Here $g_d(D)$ denotes the Gauss sum

$$g_d(D) = \sum_{\lambda\in D} e(dq(\lambda)) \tag{3.6}$$

and $g(D) = g_1(D)$.

Notice that by Milgram's formula we can evaluate $g(D)$ explicitly

$$g(D) = \sqrt{|D|}e(\operatorname{sig}(L)/8).$$

We now define vector-valued modular forms of type ρ_L. With respect to the standard basis of $\mathbb{C}[D]$ a function $f : \mathbb{H} \to \mathbb{C}[D]$ can be written in the form

$$f(\tau) = \sum_{\lambda\in D} f_\lambda(\tau)\mathfrak{e}_\lambda.$$

The following operator generalises the usual Petersson slash operator to the space of all those functions. For $\kappa \in \mathbb{Z}$ we define

$$f\mid_{\kappa,L}\gamma = j(\gamma,\tau)^{-\kappa}\rho_L(\gamma)^{-1}f(\gamma\tau), \tag{3.7}$$

where

$$j(g,\tau) = \det(g)^{-1/2}(c\tau + d)$$

is the usual automorphy factor if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$.

A holomorphic function $f : \mathbb{H} \to \mathbb{C}[D]$ is called a modular form of weight κ and type ρ_L for Γ if $f\mid_{\kappa,L}\gamma = f$ for all $\gamma \in \Gamma$, and if f is holomorphic at the cusp ∞. Here the last condition means that all Fourier coefficients $c(\lambda, n)$ with $n < 0$ vanish. If in addition $c(\lambda, n) = 0$ for all $n = 0$, we call the corresponding modular form a cusp form. We denote by $M_\kappa(\rho_L)$ the space of all such modular forms, by $S_\kappa(\rho_L)$ the subspace cusp forms. For more details see e.g. [Br1] or [BS]. Note that formula (3.4) implies that $M_\kappa(\rho_L) = \{0\}$ unless

$$2\kappa \equiv \mathrm{sig}(L) \pmod 2. \tag{3.8}$$

Therefore, if the signature of L is even, only non-trivial spaces of integral weight can occur.

The Petersson scalar product on $S_\kappa(\rho_L)$ is given by

$$(f,g) = \int_{\Gamma\backslash\mathbb{H}} \langle f(\tau), g(\tau)\rangle \,\mathrm{Im}\,\tau^\kappa d\mu(\tau) \tag{3.9}$$

where

$$d\mu(\tau) = \frac{dx\,dy}{y^2}$$

denotes the hyperbolic volume element.

In [BS] Hecke operators $T(d^2)^*$ on $S_\kappa(\rho_L)$ were introduced. More generally, for

$$\begin{aligned}\mathcal{G}(N) = \big\{&M \in \mathrm{GL}_2^+(\mathbb{Q});\quad \exists n \in \mathbb{Z} \text{ with } (n,N) = 1 \text{ such that } nM \in M_2(\mathbb{Z}) \\ &\text{and } (\det(nM), N) = 1\big\},\end{aligned}$$

$$\mathcal{Q}(N) = \big\{(M,r) \in \mathcal{G}(N) \times (\mathbb{Z}/N\mathbb{Z})^*;\quad \det(M) \equiv r^2 \pmod N\big\}$$

and $g \in \mathcal{Q}(N)$, the Hecke operator was defined in the usual way by the action of double cosets $\Gamma g \Gamma$:

$$f\mid_{\kappa,L} T(g,r) = \det(g)^{\kappa/2-1} \sum_{M \in \Gamma\backslash\Gamma g\Gamma} f\mid_{\kappa,L}(M,r) \tag{3.10}$$

In [BS] it is explained in detail how to extend ρ_L to the group $\mathcal{Q}(N)$. In this paper, we content ourselves with Hecke operators defined by matrices $g \in \mathcal{G}(\mathbb{Q})$. Let

$$\mathcal{G}^N(\mathbb{Q}) = \{g \in \mathcal{G}(\mathbb{Q}) \mid (\det(g), N) = 1\}.$$

Then $\mathcal{G}^N(\mathbb{Q})$ can be embedded homomorphically into $\mathcal{Q}(N)$ by $g \mapsto (g, t \bmod N)$, where $\det(g) = t^2$. Thus, we identify any matrix $g \in \mathcal{G}^N(\mathbb{Q})$ with $(g, t \bmod N)$ and write throughout the whole paper only $\rho_L(g)$ or $T(g)$ instead of $\rho_L(g,t)$ or $T(g,t)$, respectively.

Let $m(k,l) \in \mathcal{G}^N(\mathbb{Q}) \cap M_2(\mathbb{Z})$ with $k|l$. As already pointed out in [St1] in a similar situation, there is a close relation between the Hecke operators $T(m(k,l))$ and $T(m(k^{-1},l^{-1}))$: Clearly,

$$\Gamma m(k,l)\Gamma = \Gamma m(l,k)\Gamma = m(kl,kl)\Gamma m(k^{-1},l^{-1})\Gamma$$

and thus

$$\begin{aligned} T(m(k^{-1},l^{-1})) &= (kl)^{-(\kappa-2)}\frac{g(D)}{g_{kl}(D)}T(m(k,l)) \\ &= (kl)^{-(\kappa-2)}T(m(k,l)). \end{aligned} \tag{3.11}$$

For the last equation we have utilized that kl is a square mod N and consequently $g_{kl}(D) = g(D)$. In the same vein, we obtain

$$T(m(k,l)) = k^{\kappa-2}\frac{g_k(D)}{g(D)}T(m(\frac{l}{k},1)). \tag{3.12}$$

The factors $\frac{g(D)}{g_{kl}(D)}$ and $\frac{g_k(D)}{g(D)}$ result from the action of $m(kl,kl)$ and $m(k,k)^{-1}$, respectively in the Weil representation (cf. [BS], (3.5)). Observe that the very same definition of $\rho_L(m(k,k)^{-1})$ and $\rho_L(m(kl,kl))$ is still meaningful even if $(k,N) > 1$.

As demonstrated in [BS], Chapter 5, (3.10) defines still a Hecke operator if we assume $(k,N) > 1$ and choose $\Gamma m(k^2,1)\Gamma$ as double coset acting on $S_{k,L}$. The goal of the subsequent remarks is to extend the Hecke operator in this case to double cosets $\Gamma m(k,l)\Gamma$ and $\Gamma m(k^{-1},l^{-1})\Gamma$ with $m(k,l), m(k^{-1},l^{-1})$ as above, but $(kl,N) > 1$. We make use of the same ideas as in [St1], Chap. 4. To this end, we define

$$\rho_L^{-1}(m(k,l))\mathfrak{e}_\lambda = \frac{g_k(D)}{g(D)}\rho_L^{-1}(m(1,\frac{l}{k}))\mathfrak{e}_\lambda$$

and

$$\rho_L^{-1}(m(k^{-1},l^{-1}))\mathfrak{e}_\lambda = \frac{g(D)}{g_l(D)}\rho_L^{-1}(m(\frac{l}{k},1))\mathfrak{e}_\lambda. \tag{3.13}$$

In view of these identities, the extension to the corresponding double cosets works the same way as in [BS], Chapter 5. Consequently, (3.12) is still valid for $T(m(k^{-1},l^{-1}))$ and $T(m(k,l))$. Also, the arguments leading to (3.11) are still applicable in the case $(kl,N) > 1$. Thus, $T(m(k^{-1},l^{-1}))$ and $T(m(k,l))$ are related by this identity. However, the factor $\frac{g(D)}{g_{kl}(D)}$ is not equal to one in this case.

For the following remarks we assume that level N of L is additionally *square free*. We keep this assumption whenever we are dealing with the zeta function $\mathcal{Z}(s,f)$ (to be defined below). Let $f \in S_\kappa(\rho_L)$ be a simultaneous eigenform of all Hecke operators $T(m(k^2,1))$, $k \in \mathbb{N}$, with eigenvalues $\lambda_f(m(k^2,1))$ (see Remark 6.1, ii) in [St1] for some details when such a cusp form exists). Then the relations (3.11) and (3.12) immediately imply that f is an eigenform of all Hecke operators $T(m(k^{-1},l^{-1}))$ and $T(m(k,l))$ with eigenvalues

$$\lambda_f(m(k,l)) = k^{\kappa-2}\frac{g_k(D)}{g(D)}\lambda_f(m(\frac{l}{k},1)) \tag{3.14}$$

and

$$\lambda_f(m(k^{-1},l^{-1})) = \begin{cases} (kl)^{-(\kappa-2)}\lambda_f(m(k,l)), & (kl,N) = 1, \\ (kl)^{-(\kappa-2)}\frac{g(D)}{g_{kl}(D)}\lambda_f(m(k,l)), & (kl,N) > 1. \end{cases} \tag{3.15}$$

In [St1] the analytic properties of the *standard zeta function*

$$Z(s,f) = \sum_{k\in\mathbb{N}}\lambda_f(m(k^2,1))k^{-2s}$$

were studied. By Theorem 5.6 in [BS] this series possesses an Euler product

$$Z(s,f) = \prod_p Z_p(s,f) \tag{3.16}$$

with $Z_p(s,f) = \sum_{k\in\mathbb{N}} \lambda_f(m(p^{2k},1))p^{-2ks}$. Let us consider the more general local zeta function

$$\mathscr{Z}_p(s,f) = \sum_{(k,l)\in\Lambda_+} \lambda_f(m(p^{-k},p^{-l}))p^{-s(k+l)}.$$

By (3.15) we have

$$\mathscr{Z}_p(s,f) = \begin{cases} \sum_{(k,l)\in\Lambda_+} \lambda_f(m(p^k,p^l))p^{-(s+\kappa-2)(k+l)}, & (p,N)=1, \\ \sum_{(k,l)\in\Lambda_+} \frac{g(D)}{g_{p^{k+l}}(D)}\lambda_f(m(p^k,p^l))p^{-(s+\kappa-2)(k+l)}, & p \mid N. \end{cases} \tag{3.17}$$

Subsequently, we want to relate that the zeta functions introduced above, which boils down to evaluate the expressions $\frac{g(D)}{g_{p^{k+l}}(D)}$ and $\frac{g_k(D)}{g(D)}$ explicitly. By [We1], Lemma 5.8, we know that

$$n \mapsto \chi_D(n) = \frac{g_n(D)}{g(D)} \tag{3.18}$$

is a quadratic character of $(\mathbb{Z}/N\mathbb{Z})^\times$. More specifically,

$$\frac{g(D)}{g_n(D)} = \frac{g_n(D)}{g(D)} = \left(\frac{n}{|D|}\right) e\left(\frac{(t-1)\mathrm{oddity}(D)}{8}\right). \tag{3.19}$$

A proof for the last equation can be found in [We1], Theorem 5.17. It is known that if $|D|$ is odd, $\mathrm{oddity}(D) \equiv 0 \bmod 8$, see e. g. [We1], Lemma 5.8 or [CS], Chap. 15, § 7. Thus, $\frac{g(D)}{g_n(D)}$ simplifies to

$$\chi_D(n) = \left(\frac{n}{|D|}\right) \tag{3.20}$$

in this case. We now evaluate the quotient $\frac{g(D)}{g_{p^r}(D)}$, $r \in \mathbb{N}_0$, more explicitly for the case $p \mid N$. Decomposing D into p-groups D_p, we find

$$\frac{g(D)}{g_{p^r}(D)} = \frac{g(D_p)}{g_{p^r}(D_p)} \prod_{q\nmid N} \frac{g(D_q)}{g_{p^r}(D_q)}. \tag{3.21}$$

By the assumption on N, the level of D_p is p and therefore

$$g_{p^r}(D_p) = \begin{cases} |D_p|, & r > 0, \\ g(D_p), & r = 0. \end{cases} \tag{3.22}$$

Let $D_p^\perp$ be the orthogonal complement of D_p in D with respect to $\langle\cdot,\cdot\rangle$. Then we clearly have $D = D_p \oplus D_p^\perp$ and $(|D_p^\perp|, p) = 1$. Since $(p,q) = 1$, in view of (3.22) and Milgram's forumula the right-hand side of (3.21) can be written in the form

$$\frac{e(\mathrm{sig}(D_p)/8)}{|D_p|^{1/2}}\chi_{D_p^\perp}(p^r). \tag{3.23}$$

Notice that if p^r is a square, $g_{p^r}(D_p^\perp) = g(D_p^\perp)$ and $\chi_{D_p^\perp}(p^r) = 1$.

It follows for a "bad" prime p dividing N by means of (3.14) and (3.23) that (3.17) becomes

$$\mathcal{Z}_p(s,f) = \frac{e(\operatorname{sig}(D_p)/8)}{|D_p|^{1/2}} \sum_{l\in\mathbb{N}_0} \lambda_f(m(p^{2l},1))p^{-l(2s+k-2)} + \sum_{\substack{(k,l)\in\Lambda_+\\ k>0}} \chi_{D_p^\perp}(p^k)\lambda_f(m(p^{l-k},1)p^{-l(\kappa-2)}p^{-s(k+l)}$$

For a "good" prime $p \nmid N$ we have

$$\mathcal{Z}_p(s,f) = \sum_{(k,l)\in\Lambda_+} \chi_D(p^k)\lambda_f(m(p^{l-k},1))p^{-s(k+l)}.$$

Let χ be either of the characters χ_D, $\chi_{D_p^\perp}$. Then we can write for either of the above sums over $(k,l)\in\Lambda_+$:

$$\sum_{(k,l)} \chi(p^k)p^{-k(2s+\kappa-2)}\lambda_f(m(p^{l-k},1))p^{-(l-k)(s+\kappa-2)}.$$

For any fixed $k\in\mathbb{N}_0$ the index l runs through the set $\{2n+k \mid n\in\mathbb{N}_0\}$ to satisfy the conditions $l+k\in 2\mathbb{Z}$ and $l\geq k$. Therefore, the index $l-k$ runs through

$$\{2(n-k) \mid n\in\mathbb{N}_0 \text{ with } n\geq k\} = 2\mathbb{N}_0.$$

Thus, we may rewrite the latter series as

$$\sum_k \chi(p^k)p^{-2ks}\sum_n \lambda_f(m(p^{2n},1))p^{-2ns}.$$

Globally, we then have

(3.24)

$$\mathcal{Z}(s,f) = \prod_{p\mid\mid D\mid} \left(\left(\frac{e(\operatorname{sig}(D_p)/8)}{|D_p|^{1/2}}-1\right) + L_p(2s+\kappa-2,\chi_{D_p^\perp})\right) L(2s+\kappa-2,\chi_D)Z(s+\kappa-2,f),$$

where

i)
$$L_p(s,\chi_{D_p^\perp}) = (1-\chi_{D_p^\perp}(p)p^{-s})^{-1},$$

ii) $L(s,\chi_D)$ is the Dirichlet L-series associated to χ_D.

We can now state the following theorem regarding the analytic properties of $\mathcal{Z}(s,f)$.

Theorem 3.3. *Let $\kappa\in 2\mathbb{Z}$, $\kappa\geq 3$, satisfy $2\kappa+\operatorname{sig}(L)\equiv 0 \bmod 4$ and $f\in S_\kappa(\rho_L)$ a common eigenform of all Hecke operators $T(m(k^2,1))$, $k\in\mathbb{N}$. Then the zeta function $\mathcal{Z}(s,f)$ has a meromorphic continuation to the whole s-plane.*

Proof. This results from (3.24): Theorem 6.6 of [St1] provides the desired property for $Z(s,f)$. For $L(s,\chi_D)$ this is well known and clear for the remaining factor anyway. □

4. The Weil representation on $\mathrm{GL}_2(\mathbb{A})$

Let $(L,(\cdot,\cdot))$ and $D=L'/L$ be defined as in Section 3 We further define $V=L\otimes\mathbb{Q}$ and let $H=O(V)$ be the orthogonal group over $\mathbb{Q}$ attached to $(V,(\cdot,\cdot))$. In this section we collect some well known facts on the Weil representation of $\mathrm{SL}_2(\mathbb{A})\times H(\mathbb{A})$, which is suited for our purposes in this paper. Here, we consider the Schrödinger model of the Weil repesentation $\omega = \prod_{p\leq\infty}\omega_p$ on the space $S(V(\mathbb{A}))$ of Schwartz-Bruhat functions associated to the character

$$\psi = \prod_{p\leq\infty}\psi_p : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times,\ x = (x_p) \mapsto \psi(x) = e^{2\pi i(-x_\infty + \sum_{p<\infty} x'_p)}, \tag{4.1}$$

where $x'_p \in \mathbb{Q}/\mathbb{Z}$ is the principal part of x_p and $V(\mathbb{A}) = V \otimes \mathbb{A}$. Note that this character is the complex conjugate of the standard additive character (see e. g. [St], [BY] and [KL], Chapter 8).

A second goal of present section is the extension of ω to a subgroup of $\mathrm{GL}_2(\mathbb{A})$ in the spirit of the extension of the "finite" Weil representation ρ_L in [BS].

For $\mu \in D$ we define $\varphi_\mu \in S(V(\mathbb{A}_f))$ with

$$\varphi_\mu = \mathbb{1}_{\mu+\hat{L}} = \prod_{p<\infty}\varphi_p^{(\mu)} = \prod_{p<\infty}\mathbb{1}_{\mu+L_p}. \tag{4.2}$$

Here $L_p = L \otimes \mathbb{Z}_p$, which is the p-part of $\hat{L} = L \otimes \hat{\mathbb{Z}}$ with $\hat{\mathbb{Z}} = \prod_{p<\infty}\mathbb{Z}_p$, and $\mathbb{1}_{\mu+L_p}$ is the characteristic function of $\mu + L_p$. Note that there is a close relation between the finite groups L'_p/L_p and the p-groups D_p. In fact, these groups are isomorphic. This isomorphism additionally respects the quadratic forms, which endow both groups, see e. g. [Ze], Section 3, [St], Remark 3.2 or [We1], Theorem 4.30. In the following, we identify these groups and use them interchangeably. As in [BY] we consider the $|D|$- dimensional subspace

$$S_L = \bigoplus_{\mu\in D}\mathbb{C}\varphi_\mu \subset S(V(\mathbb{A}_f)). \tag{4.3}$$

It is known that the space (4.3) is stable under the action of the group $\mathrm{SL}(2,\hat{\mathbb{Z}})$ via the Weil representation ω_f (see e. g. [BY], chapter 2 or [Ku]). Also, the L^2 scalar product $\langle\cdot,\cdot\rangle$ on $S_L \subset S(V(\mathbb{A}_f))$ simplifies to

$$\langle\sum_{\mu\in D}F_\mu\varphi_\mu, \sum_{\mu\in D}F_\mu\varphi_\mu\rangle = \sum_{\mu\in D}|F_\mu|^2. \tag{4.4}$$

Note that D can be decomposed into p-groups $D = \bigoplus_{p||D|} D_p \cong \bigoplus_{p||D|} L'_p/L_p$. For almost all primes p - those coprime to $|D|$ - L_p is unimodular and thus $L'_p/L_p = 0 + L_p$. Therefore, we can write $D \cong \bigoplus_{p<\infty} L'_p/L_p$. On the level of the space S_L, this decomposition translates to the isomorphism

$$S_L \cong \bigotimes_{p<\infty} S_{L_p}, \quad \varphi_\mu \mapsto \bigotimes_{p<\infty}\varphi_p^{(\mu_p)}, \tag{4.5}$$

where $\mu = \sum_{p||D|}\mu_p$ and $\varphi_p^{(\mu_p)} = \varphi_p^{(0)}$ for all primes p coprime to $|D|$. The local Weil representation ω_p acts on the p-part S_{L_p} of S_L, where

$$S_{L_p} = \begin{cases} \bigoplus_{\mu\in L'_p/L_p}\mathbb{C}\varphi_p^{(\mu)}, & p \mid |D|, \\ \mathbb{C}\varphi_p^{(0)}, & p \nmid |D|. \end{cases} \tag{4.6}$$

We then have

$$\omega_f(\gamma_f)\varphi_\mu = \bigotimes_{p<\infty}\omega_p(\gamma_p)\varphi_p^{(\mu_p)}.$$

According to [St], Lemma 3.4 and [BY], Proposition 2.5, the Weil representation ω_p can be described explicitly on the generators of $\mathrm{SL}_2(\mathbb{Z}_p)$ by

$$\begin{aligned} \omega_p(n(b))\varphi_p^{(\mu)} &= \psi_p(bq(\mu))\varphi_p^{(\mu)} \\ \omega_p(w)\varphi_p^{(\mu)} &= \frac{\gamma_p(L'_p/L_p)}{|L'_p/L_p|^{1/2}} \sum_{\nu_p\in L'_p/L_p} \psi_p((\mu_p,\nu_p))\varphi_p^{(\nu_p)} \\ \omega_p(m(a))\varphi_p^{(\mu_p)} &= \chi_{V,p}(a)\varphi_p^{(a^{-1}\mu_p)}, \end{aligned} \tag{4.7}$$

where $\gamma(L'_p/L_p)$ is the local Weil index and $\chi_{V,p}(a) = (a,(-1)^{m/2}|D_p|)_p$ is the local Hilbert symbol. Evaluating the local Hilbert symbol gives

$$\chi_{V,p}(a) = \left(\frac{a}{|L'_p/L_p|}\right) = \chi_{D_p}(a), \tag{4.8}$$

see e. g. [Se], Chapter III. These formulas imply that (see the proof Lemma 3.4 in [St])

i) the local Weil representations ω_p is trivial if p is coprime to $|D|$,
ii) if we identify $\mathbb{C}[D]$ with S_L via $\mathfrak{e}_\mu \mapsto \varphi_\mu$, then ω_f coincides with the finite Weil representation $\rho_{L.1}$ in the following way

$$\rho_L(\gamma) = \omega_f(\gamma_f), \tag{4.9}$$

where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\gamma_f \in \mathrm{SL}_2(\widehat{Z})$ is the projection of γ into $\mathrm{SL}_2(\widehat{\mathbb{Z}})$. Note that by our choice of the character ψ, the relation (4.9) differs from the one in [BY], (2.7), by conjugation.

The following lemma provides the action of ω_p for the lower triangular matrix $n_-(c) \in \mathrm{SL}_2(\mathbb{Z}_p)$:

Lemma 4.1. *Let $c \in \mathbb{Z}_p^\times$. Then*

$$\omega_p(n_-(c))\varphi_p^{(\mu_p)} = \frac{\gamma_p(L'_p/L_p)}{|L'_p/L_p|^{1/2}}\chi_{V,p}(-c)\psi_p(c^{-1}q(\mu_p)) \sum_{\nu_p\in L'_p/L_p} \psi_p(c^{-1}q(\nu_p))\psi_p(-c(\mu_p,\nu_p))\varphi_p^{(\nu_p)}. \tag{4.10}$$

Proof. By the Bruhat decomposition (see e. g. [KL], p. 69),

$$n_-(c) = n(c^{-1})wn(c)m(-c).$$

From this we infer that by means of (4.7)

$$\omega_p(n_-(c))\varphi_p^{(\mu_p)} = \frac{\gamma_p(L'_p/L_p)}{|L'_p/L_p|^{1/2}}\chi_{V,p}(-c)\psi_p(c^{-1}q(\mu_p)) \sum_{\nu_p\in L'_p/L_p} \psi_p(c^{-1}q(\nu_p))\psi_p(-c(\mu_p,\nu_p))\varphi_p^{(\nu_p)}.$$

□

Via the extension of ρ_L to a subgroup of $\mathcal{G}(\mathbb{Q})$ (see Section 3) it is possible to extend ω_f (cf. [We], Def. 46) to the group $\mathcal{K} = \prod_{p<\infty}\mathcal{K}_p$ as follows:

Definition 4.2. Let $N = \prod_{i=1}^r p^{e_i}$ be the level of D,

$$Q(N) = \{M \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \mid \det(M) \text{ a square in } (\mathbb{Z}/N\mathbb{Z})^\times\}$$

(cf. [BS], (3.2)) and

$$\mathcal{K} \xrightarrow{\pi} \prod_{p|N}\mathcal{K}_p \xrightarrow{\pi_N} Q(N),$$

where π is the projection onto the places $p \mid N$ and π_N is the composition of the canonical projection of $\mathbb{Z}_{p_i}$ to $\mathbb{Z}/p^{e_i}\mathbb{Z}$ and the application of the Chinese remainder theorem, applied to each component of the matrices. For $k \in \mathcal{K}$ we then define

$$\begin{aligned} \omega_f(k) &= \bigotimes_{p<\infty} \omega_p(k_p) \\ &= \bigotimes_{p\nmid N} \omega_p(k_p) \bigotimes_{p\mid N} \omega_p(k_p) \end{aligned} \tag{4.11}$$

with $\omega_p(k_p) = \mathrm{id}_{S_{L_p}}$ for all primes $p \nmid N$ and

$$\bigotimes_{p\mid N} \omega_p(k_p) = \rho_L(\pi_N((k_p)_{p\mid N})). \tag{4.12}$$

Here by $(k_p)_{p\mid N}$ we mean the tuple of all components $k_p \in \mathcal{K}_p$ of k belonging to the primes p dividing N. Combining (4.11) and (4.12) we can write

$$\omega_f(k) = \rho_L((\pi \circ \pi_N)(k)). \tag{4.13}$$

Note that Definition 4.2 is compatible with (4.9). For, if we take $k \in \mathrm{SL}_2(\widehat{\mathbb{Z}})$ as the projection of some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we find $(\pi_N \circ \pi)(k) = \pi_N(\gamma) \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and $\omega_f(k) = \rho_L(\gamma)$ since ρ_L factors through $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

We can imitate the extension process of ρ_L from Γ to $\mathcal{Q}(N)$ to extend the local Weil representation ω_p from $\mathrm{SL}_2(\mathbb{Z}_p)$ to $\mathcal{K}_p$.

Definition 4.3. Let $k_p \in \mathcal{K}_p$ with $\det(k_p) = t^2 \in (\mathbb{Z}_p)^\times$. Then we define in accordance with [BS], (3.5),

$$\begin{aligned} \omega_p((\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}))\varphi_p^{(\lambda_p)} &= \frac{g(D_p)}{g_t(D_p)}\varphi_p^{(\lambda_p)} \\ &= \chi_{D_p}(t)\varphi_p^{(\lambda_p)} \end{aligned} \tag{4.14}$$

and

$$\omega_p(k_p) = \omega_p((\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}))\omega_p(\begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix} k_p), \tag{4.15}$$

where $\begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix} k_p$ is an element of $\mathrm{SL}_2(\mathbb{Z}_p)$.

If the level N is equal to a prime p, we have $\omega_f = \omega_p$ and Definition 4.2 specialises to an extension of the local Weil representation ω_p to $\mathcal{K}_p$. Thus, $\omega_p(k_p) = \rho_L(\pi_p(k_p))$, which is compatible with Definition 4.3.

Finally, we need to define the local Weil representation ω_p on double cosets of the form $\mathcal{K}_p m(p^{-k}, p^{-l})\mathcal{K}_p$. Here $m(p^{-k}, p^{-l}) \in \mathcal{G}(\mathbb{Q})$ and $k \leq l$ (cf. Theorem 6.9). Again, we mirror the corresponding process in [BS], Chapter 5, and make use of results in Section 3 of the present paper.

Definition 4.4. Let $m(p^{-k}, p^{-l}) \in \mathcal{G}(\mathbb{Q})$ with $k \leq l$.

i) Then we put

$$\begin{aligned} \omega_p^{-1}(m(p^{-k}, p^{-l}))\varphi_p^{(\lambda_p)} &= \omega_p^{-1}(m(p^{-l}, p^{-l}))\omega_p^{-1}(m(p^{l-k}, 1)\varphi_p^{(\lambda_p)} \\ &= \frac{g(D_p)}{g_{p^l}(D_p)}\varphi_p^{(p^{(l-k)/2}\lambda_p)}. \end{aligned} \tag{4.16}$$

ii) For $\delta = \gamma m(p^{-k}, p^{-l})\gamma' \in \mathcal{K}_p m(p^{-k}, p^{-l})\mathcal{K}_p$ we define

$$\omega_p^{-1}(\delta)\varphi_p^{(\lambda_p)} = \omega_p^{-1}(\gamma')\omega_p^{-1}(m(p^{-k}, p^{-l}))\omega_p^{-1}(\gamma)\varphi_p^{(\lambda_p)}. \tag{4.17}$$

As already noted before, if the level N is equal to a prime p, we have $\omega_p(\gamma_p) = \omega_f(\gamma_p) = \rho_L(\pi_p(\gamma_p))$ for $\gamma_p \in \mathrm{SL}_2(\mathbb{Z}_p)$. In view of (3.13), we conclude $\omega_p^{-1}(\delta) = \rho_L^{-1}(\delta)$ for $\delta \in \mathrm{SL}_2(\mathbb{Z}_p)m(p^{-k}, p^{-l})\,\mathrm{SL}_2(\mathbb{Z}_p)$. Proposition 5.1 in [BS] then shows that (4.17) is independent of the choice of $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z}_p)$. Since the action of ω_p on $\mathcal{K}_p$ differs from that on $\mathrm{SL}_2(\mathbb{Z}_p)$ only by constant factor (see Definition 4.3), we obtain the same result for $\gamma, \gamma' \in \mathcal{K}_p$.

5. The Hecke algebra $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$

In this section we will describe the structure of the local vector valued spherical Hecke algebra $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ associated to the pair of groups $(\mathcal{Q}_p, \mathcal{K}_p)$ and the local Weil representation ω_p. For each prime p we will introduce a Satake map, which allows us to understand the structure of the corresponding Hecke algebra. For primes $p \nmid |D|$ the Hecke algebras are isomorphic to the scalar valued algebras defined by the same groups. These are well understood thanks to the classical Satake map. If p divides $|D|$, the algebras $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ are considerably more complicated because ω_p is non-trivial. However, under certain restrictions for D_p, we will define a modified Satake map, which maps $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ to a simpler algebra, whose structure can be easier determined.

The following general facts about spherical Hecke algebras can be found in many places among them [BK], chapter 4, [Ho] and [Mu].

Definition 5.1. Let G be a locally compact group G, K an open compact subgroup and $\rho : K \to \mathrm{GL}(V)$ a representation of K. The Hecke algebra $\mathcal{H}(G//K, \rho)$ of ρ-spherical functions is the set of functions $f : G \to \mathrm{End}(V)$ which are

i) compactly supported modulo K, i. e. each f vanishes outside finitely many double cosets KgK and satisfy
ii)
$$f(k_1 g k_2) = \rho(k_1) \circ f(g) \circ \rho(k_2) \text{ for all } k_1, k_2 \in K \text{ and all } g \in G.$$

Since each element f of $\mathcal{H}(G//K, \rho)$ is of the form

$$f(g) = \sum_{i=1}^{n} a_i(f_i)(g),$$

where f_i is an element of the subspace of functions of $\mathcal{H}(G//K, \rho)$, which vanish outside Kg_iK, the whole algebra is generated by the functions f_i. Similarly, we denote by $\mathcal{H}(G//K)$ the set of functions $f : G \to \mathbb{C}$, which are compactly supported modulo K and K-bi-invariant, i. e. $f(k_1 g k_2) = f(g)$ for all $k_1, k_2 \in K$ and all $g \in G$. We call $\mathcal{H}(G//K)$ also a Hecke algebra.

It is well known that $\mathcal{H}(G//K, \rho)$ is an associative $\mathbb{C}$-algebra with respect to convolution

$$(f_1 * f_2)(g) = \int_G f_1(h) \circ f_2(h^{-1}g)dh = \sum_{h \in G/K} f_1(h) \circ f_2(h^{-1}g), \tag{5.1}$$

where dh is the standard Haar measure on G normalized by $\int_K dh = 1$

In order determine the structure of $\mathcal{H}(G//K, \rho)$, in view of the remarks before, it is useful to study the space of functions in this Hecke algebra, which vanish outside a single double coset KgK. It can be described in terms of intertwining operators of ρ associated with g. To state the corresponding result, we fix some notation. For $g \in G$ we mean by K^g the group gKg^{-1} and write ρ_g for the representation $h \mapsto \rho_g(h) = \rho(g^{-1}hg)$ of K^g. As usual,

$$\mathrm{Hom}_{K\cap K^g}(\rho, \rho_g) = \{F : V \to V \mid F \text{ is linear and } F \circ \rho_g(h) = \rho(h) \circ F \text{ for all } h \in K \cap K^g\}.$$

Then we have

Lemma 5.2. *Let $g \in G$. The subspace of $\mathcal{H}(G//K, \rho)$ consisting of functions supported on KgK, is isomorphic to* $\mathrm{Hom}_{K\cap K^g}(\rho,\ \rho_g)$*.*

Proof. The assertion is well known (see e. g. [BK], Chapter 4). Nevertheless, for later purposes, we indicate a proof by giving the maps of the claimed isomorphism (without further explanation).

If $f \in \mathcal{H}(G//K, \rho)$ with $f(g) \neq 0$ supported on KgK, then it easily checked that $f(g) \in \mathrm{Hom}_{K\cap K^g}(\rho,\ \rho_g)$ (non-zero). On the other hand, if $0 \neq F \in \mathrm{Hom}_{K\cap K^g}(\rho,\ \rho_g)$, we put $f(g) = F$ and $f(k_1gk_2) = \rho(k_1) \circ f \circ \rho(k_2)$ and obtain thereby an element of the above stated subspace of $\mathcal{H}(G//K, \rho)$. □

The following Lemma ensures that the groups $\mathcal{Q}_p$ and $\mathcal{K}_p$ meet the conditions of Definition 5.1. It might be known. Since I have not found it in the literature, I state it here and add a short proof.

Lemma 5.3. i) *The group $\mathcal{K}_p$ is an open compact subgroup of $\mathcal{Q}_p$.*
ii) *The group $\mathcal{Q}_p$ is a locally compact subgroup of* $\mathrm{GL}_2(\mathbb{Q}_p)$*.*

Proof. It is well known that $\mathrm{GL}_2(\mathbb{Q}_p)$ is locally compact. By Lemma 8, I.3, of [Ch], it follows that $\mathcal{Q}_p$ is also locally compact. By [Ka], Thm. 2.15, we know that $(\mathbb{Z}_p^\times)^2$ is an open subgroup in $\mathbb{Z}_p^\times$ Therefore, $\mathcal{K}_p$ is an open subgroup in $\mathrm{GL}_2(\mathbb{Z}_p)$, which implies that is is also closed (see [HW], Thm. 5.5). As $\mathcal{Q}_p \cap \mathrm{GL}_2(\mathbb{Z}_p) = \mathcal{K}_p$, we find that $\mathcal{K}_p$ is a compact subgroup of $\mathcal{Q}_p$. □

Note that there is analogue of the Cartan decomposition for the pair $(\mathcal{Q}_p, \mathcal{K}_p)$.

Lemma 5.4. *The group $\mathcal{Q}_p$ can be written as a disjoint union of $\mathcal{K}_p$-double cosets:*

$$\mathcal{Q}_p = \bigcup_{\substack{k\leq l\\ k+l\in 2\mathbb{Z}}} \mathcal{K}_p m(p^k, p^l)\mathcal{K}_p.$$

Proof. The proof is the same as for the Cartan decomposition for $\mathrm{GL}_2(\mathbb{Q}_p)$, see e. g. [Mu], p. 17. In the quoted proof the matrix $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_p)$ with $a \neq 0$ and $|a|_p \geq \max\{|b|_p, |c|_p, |d|_p\}$ is transformed into

$$m(a, d - a^{-1}bc) = k_3gk_4,$$

where $k_3 = n_{\text{-}}(-a^{-1}c), k_4 = n(-a^{-1}b) \in \mathcal{K}_p$. If we assume that $\det(g) = p^{2x}y^2$ and $a = p^k s$, then $d - a^{-1}bc = p^{2x-k}y^2s^{-1}, y, s \in \mathbb{Z}_p^\times$, and

$$m(p^k, p^{2x-k}) = n_{\text{-}}(-a^{-1}c)g\ n(-a^{-1}b)m(s)m(1, y^2).$$

Therefore, all used transformation matrices are contained in $\mathcal{K}_p$. We have a similar decomposition if $d \neq 0$, see [KL], p. 208.

Also, note that any two double cosets $\mathcal{K}_pg_1\mathcal{K}_p$, $\mathcal{K}_pg_2\mathcal{K}_p$ are disjoint since otherwise the double cosets $\mathrm{GL}_2(\mathbb{Z}_p)g_1\,\mathrm{GL}_2(\mathbb{Z}_p)$, $\mathrm{GL}_2(\mathbb{Z}_p)g_2\,\mathrm{GL}_2(\mathbb{Z}_p)$ would not be disjoint, contradicting the Cartan decomposition for $\mathrm{GL}_2(\mathbb{Q}_p)$. □

We also have an analogue of the Iwasawa decomposition of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Lemma 5.5.

$$\mathcal{Q}_p = (B(\mathbb{Q}_p) \cap \mathcal{Q}_p)\mathcal{K}_p. \tag{5.2}$$

Proof. This follows immediately from the Iwasawa decomposition for $\mathrm{GL}_2(\mathbb{Q}_p)$ by the intersection with $\mathcal{Q}_p$ on both sides. □

As already noted, there are two cases to consider regarding the structure of $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$. It depends on whether p divides $|D|$ or not. In both cases we will determine a set of generators with the help of Lemma 5.2. Afterwards, we will define a *Satake map*. If p divides $|D|$, we will show - under some restrictions on L_p'/L_p - that $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ is isomorphic to a subalgebra of the spherical Hecke algebra $\mathcal{H}(\mathcal{M}_p//\mathcal{D}_p, \omega_{p_{|S_{L_p}^{N(\mathbb{Z}_p)}}})$, where

$$S_{L_p}^{N(\mathbb{Z}_p)} = \{\varphi \in S_{L_p} \mid \omega_p(n)(\varphi) = \varphi \text{ for all } n \in N(\mathbb{Z}_p)\}.$$

If $(p, |D|) = 1$, $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ is isomorphic to the classical spherical Hecke algebra $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p)$, whose structure is well known. We start with the discussion of the first mentioned case and consider the latter case subsequently.

To describe the structure of $\mathrm{Hom}_{\mathcal{K}_p \cap \mathcal{K}_p^g}(\omega_p, \rho_g)$, we need the decomposition of S_{L_p} into irreducible submodules. This decomposition is well known, see for example [NW], Satz 2, Satz 4 and pages 521-522. We recall those parts relevant for next lemma. We denote with $\mathrm{Aut}(D)$ the group of all automorphisms ε of D satisfying $q(\varepsilon(x)) = q(x)$ for all $x \in D$. Let further U be a subgroup of $\mathrm{Aut}(D)$ and $\widehat{U}$ the dual group of U. It turns out that most of the *primitive* characters in $\widehat{U}$ give rise to irreducible represenation. The definition of a primitive character can be found on page 491 in [NW]. We have to distinguish between the two possible anisotropic modules. For the case $\mathcal{A}_p^t \oplus \mathcal{A}_p^1$ Nobs and Wolfart proved the following decomposition of the space S_{L_p} with respect to the Weil representation

$$S_{L_p} \cong S_{L_p}(\chi_1) \bigoplus_{\substack{\chi \in \widehat{U} \text{ primitiv} \\ \chi^2 \neq 1}} S_{L_p}(\chi) \oplus \left(S_{L_p}(1,-) \oplus S_{L_p}(t,-)\right), \tag{5.3}$$

where $\chi_1 = 1$ means the trivial character and

$$\begin{aligned} S_{L_p}(\chi) &= \left\{ f \in S_{L_p} \mid f(\varepsilon x) = \chi(\varepsilon) f(x) \text{ for all } x \in \mathcal{A}_p^t \oplus \mathcal{A}_p^1 \text{ and all } \varepsilon \in U \right\}, \\ S_{L_p}(t,-) &= \left\{ f \in S_{L_p} \mid f(-x) = -f(x) \text{ for all } x \in \mathcal{A}_p^t \right\}. \end{aligned} \tag{5.4}$$

The space $S_{L_p}(1,-)$ is defined the same way by simply replacing t with 1. We write

$$f = f_1 + \sum_{\substack{\chi \in \widehat{U} \text{ primitiv} \\ \chi^2 \neq 1}} f_\chi + f_+ + f_- \tag{5.5}$$

for an element in S_{L_p} with respect to (5.4). It is shown in [NW] that $S_{L_p}(\chi_1)$ and $S_{L_p}(\chi_2)$ are isomorphic if and only if $\chi_1 = \chi_2$ or $\chi_1 = \overline{\chi_2}$. The remaining modules in (5.3) are (pairwise) not isomorphic. The isomorphism between $S_{L_p}(\chi)$ and $S_{L_p}(\overline{\chi})$ is given explicitely in terms of the generators of $S_{L_p}(\chi)$: A generator

$$f_{\mu_p}(\chi) = \sum_{\varepsilon \in U} \chi(\varepsilon) \varphi_p^{\varepsilon(\mu_p)} \tag{5.6}$$

is mapped to $f_{\overline{\mu_p}}(\overline{\chi})$, where $\overline{\mu_p}$ is $(a+b,-b)$ for $\mu_p=(a,b)$. We denote this intertwining operator by $T^{\overline{\chi}}$.

5.1. **The case of primes p dividing $|D|$.**

Lemma 5.6. *Let D_p be an anisotropic discriminant form and $g=m(p^k,p^l)\in\mathcal{Q}_p$. Put $\rho_g=(\omega_p)_g$.*

i) *If $k<l$, then the space $\mathrm{Hom}_{\mathcal{K}_p\cap\mathcal{K}_p^g}(\omega_p,\rho_g)$ is generated by the map*

$$T(k,l):S_{L_p}\to S_{L_p}^{N(\mathbb{Z}_p)},\quad \varphi_p^{(\mu)}\mapsto T(k,l)(\varphi_p^{(\mu)})=\frac{g(D_p)}{g_{p^l}(D_p)}\varphi_p^{(p^{(l-k)/2}\mu_p)}=\frac{g(D_p)}{g_{p^l}(D_p)}\varphi_p^{(0)}. \tag{5.7}$$

ii) *If $D_p\cong\mathcal{A}_p^t\oplus\mathcal{A}_p^1$, then S_{L_p} decomposes into the irreducible submodules For $k=l$ the space $\mathrm{Hom}_{\mathcal{K}_p\cap\mathcal{K}_p^g}(\omega_p,\rho_g)$ is then generated by the maps $T(k,k)^{\chi}$, $T(k,k)^{\overline{\chi}}$, $T(k,k)^{\chi_1}$, $T(k,k)^+$ and $T(k,k)^-$ where*

$$\begin{aligned}
&T(k,k)^{\chi}(f)=\frac{g(D_p)}{g_{p^k}(D_p)}f_{\chi},\quad T(k,k)^{\overline{\chi}}(f)=\frac{g(D_p)}{g_{p^k}(D_p)}T^{\overline{\chi}}(f_{\chi}),\\
&T(k,k)^{+}(f)=\frac{g(D_p)}{g_{p^k}(D_p)}f_{+},\quad T(k,k)^{-}(f)=\frac{g(D_p)}{g_{p^k}(D_p)}f_{-}\ \text{and}\\
&T(k,k)^{\chi_1}(f)=\frac{g(D_p)}{g_{p^k}(D_p)}f_{1}
\end{aligned}\tag{5.8}$$

and f is an element in S_{L_p} as in (5.5).

iii) *If $D_p\cong\mathcal{A}_p^t$, then S_{L_p} decomposes into the irreducible submodules $S_{L_p}(t,+)$ and $S_{L_p}(t,-)$. For $k=l$ the space $\mathrm{Hom}_{\mathcal{K}_p\cap\mathcal{K}_p^g}(\omega_p,\rho_g)$ is then generated by the two maps $T(k,k)^+$ and $T(k,k)^-$, where*

$$T(k,k)^{+}(f_++f_-)=\frac{g(D_p)}{g_{p^k}(D_p)}f_{+}\ \text{and}\ T(k,k)^{-}(f)=\frac{g(D_p)}{g_{p^k}(D_p)}f_{-}. \tag{5.9}$$

Proof. In light of Lemma 5.4, it clearly suffices to choose $g=m(p^k,p^l)$ with $k\le l$.

i) First, note that

$$m(p^k,p^l)\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)m(p^k,p^l)^{-1}=\left(\begin{smallmatrix}a&bp^{k-l}\\cp^{l-k}&d\end{smallmatrix}\right).$$

In particular, for $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=n(p^{l-k})$ we find that $n(1)$ is an element of $\mathcal{K}_p\cap\mathcal{K}_p^g$. Thus, for any $F\in\mathrm{Hom}_{\mathcal{K}_p\cap\mathcal{K}_p^g}(\omega_p,\rho_g)$ the equation

$$\begin{aligned}
F\circ\omega_p(m(p^k,p^l)^{-1}n(1)m(p^k,p^l))=\omega_p(n(1))\circ F &\Longleftrightarrow \omega_p(n(1))\circ F=F\circ\omega_p(n(p^{l-k}))\\
&\Longleftrightarrow \omega_p(n(1))\circ F=F
\end{aligned}$$

must hold. For the last equivalence we have used that the level of L_p'/L_p is p. It follows that the image of F is a subset of $S_{L_p}^{N(\mathbb{Z}_p)}$. Since the identity $\omega_p(n(b))\varphi_p^{(\gamma)}=\varphi_p^{(\gamma)}$ holds for all $b\in\mathbb{Z}_p^\times$ if and only if γ is isotropic and L_p'/L_p anisotropic, we can conclude that $S_{L_p}^{N(\mathbb{Z}_p)}=\mathbb{C}\varphi_p^{(0)}$. Therefore, F is a scalar multiple of the map $\varphi_p^{(\lambda_p)}\mapsto\varphi_p^{(0)}$ and has the claimed form.

ii) The decomposition of S_{L_p} into irreducible submodules is well known. For the quadratic module $\mathcal{A}_p^t$ see for example [NW], Theorem 4. The case of the quadratic module $\mathcal{A}_p^t\oplus\mathcal{A}_p^1$

is treated in [NW], Theorem 2 and Section 9, p. 521-522. In the case $k = l$ the equation $F \circ \rho_g(h) = \omega_p(h) \circ F$ simplifies to $F \circ \omega_p(h) = \omega_p(h) \circ F$ for all $h \in \mathcal{K}_p$. Thus, F is an intertwining operator for ω_p. The structure of the space of intertwining operators can be found in books about representation theory, cf. [JL], Chapter 11. □

Remark 5.7. The factor $\frac{g(D_p)}{g_{p^l}(D_p)}$ is not necessary. We introduced it for reasons of compatibility (see Theorem 6.9 for that matter).

In view of Lemma 5.2 and Lemma 5.6, the following corollary is immediate.

Corollary 5.8. *Let p be a prime dividing $|D|$ and D_p anisotropic. Then the Hecke algebra $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ is generated by the following elements:*

i) *For $k < l$*

$$T_{k,l}(k_1 m(p^k, p^l) k_2) = \omega_p(k_1) \circ T(k,l) \circ \omega_p(k_2), \tag{5.10}$$

where $T_{k,l}$ is only supported on $\mathcal{K}_p m(p^k, p^l)\mathcal{K}_p$. Here $T(k,l)$ is the intertwining operator specified in Lemma 5.6, i).

ii) *For $k = l$*

$$T_{k,k}(k_1 m(p^k, p^k) k_2) = \omega_p(k_1) \circ T(k,k) \circ \omega_p(k_2), \tag{5.11}$$

where $\operatorname{supp}(T^{(i)}_{k,k}) = \mathcal{K}_p m(p^k, p^k)\mathcal{K}_p$ and $T(k,k)$ is one of the operators

$$T(k,k)^{\chi}, T(k,k)^{\overline{\chi}}, T(k,k)^{+}, T(k,k)^{-} \text{ or } T(k,k)^{\chi_1}$$

given in Lemma 5.6, ii).

The following theorem investigates the structure of the Hecke algebra $\mathcal{H}(\mathcal{M}_p//\mathcal{D}_p, \omega_{p}{}_{|S_{L_p}^{N(\mathbb{Z}_p)}})$ assuming D_p is anisotropic.

Theorem 5.9. *Let p be an odd prime dividing $|D|$ and D_p anisotropic.*

i) *Then*

$$\begin{aligned} \omega_p(m(t_1,t_2))\varphi_p^{(0)} &= \begin{cases} \left(\frac{t_2}{p}\right)\varphi_p^{(0)}, & |D_p| = p, \\ \varphi_p^{(0)}, & |D_p| = p^2 \end{cases} \\ &= \chi_{D_p}(t_2)\varphi_p^{(0)} \end{aligned} \tag{5.12}$$

for all $m(t_1,t_2) \in \mathcal{D}_p$.

ii) *Then $S_{L_p}^{N(\mathbb{Z}_p)}$ is equal to $\mathbb{C}\varphi_p^{(0)}$ and the Hecke algebra $\mathcal{H}(\mathcal{M}_p//\mathcal{D}_p, \omega_{p}{}_{|S_{L_p}^{N(\mathbb{Z}_p)}})$ isomorphic to the scalar valued Hecke algebra $\mathcal{H}(\mathcal{M}_p//\mathcal{D}_p)$.*

Proof. As already mentioned in Section 3, the order of an isotropic quadratic module D_p is either p^2 or p.

i) Let $m(t_1,t_2) \in \mathcal{D}_p$ with $\det(m(t_1,t_2)) = t^2 \in (\mathbb{Z}_p^\times)^2$. Then by (4.15)

$$\begin{aligned} \omega_p(m(t_1,t_2))\varphi_p^{(0)} &= \omega_p(m(t,t))\omega_p(m(t^{-1}t_1, t^{-1}t_2))\varphi_p^{(0)} \\ &= \left(\frac{t}{|L_p'/L_p|}\right)\left(\frac{t^{-1}t_2}{|L_p'/L_p|}\right)\varphi_p^{(0)}, \end{aligned}$$

where we have used (4.8) and (4.14). The claimed result now follows.

For ii) we first note that if L_p'/L_p is anisotropic, the space $S_{L_p}^{N(\mathbb{Z}_p)}$ is equal to $\mathbb{C}\varphi_p^{(0)}$ and thus one-dimensional. From i) we know that $\mathcal{D}_p$ acts via ω_p on $S_{L_p}^{N(\mathbb{Z}_p)}$ by multiplication with the quadratic character χ_{D_p}. Consequently,

$$\mathcal{H}(\mathcal{M}_p//\mathcal{D}_p, {\omega_p}_{|S_{L_p}^{N(\mathbb{Z}_p)}}) = \mathcal{H}(\mathcal{M}_p//\mathcal{D}_p, \chi_{D_p}),$$

where the latter Hecke algebra is meant in the sense of Definition 5.1 with the one-dimensional representation $\rho = \chi_{D_p}$. The structure of the latter algebra was discussed in [Ho], Remark 5.1. It was stated there that $\mathcal{H}(\mathcal{M}_p//\mathcal{D}_p, \chi_{D_p})$ is isomorphic to the usual spherical algebra $\mathcal{H}(\mathcal{M}_p//\mathcal{D}_p)$. To state this isomorphism explicitly, we specify a set of generators for the former algebra. It is generated by elements of the form

$$T_{k,l}(m(t_1,t_2)m(p^k,p^l)m(s_1,s_2)) = \chi_{D_p}(m(t_1,t_2)) \circ T(k,l) \circ \chi_{D_p}(m(s_1,s_2))$$

with

$$T(k,l) = \mathbb{1}_{\mathcal{D}_p m(p^k,p^l)\mathcal{D}_p} \operatorname{id}_{S_{L_p}^{N(\mathbb{Z}_p)}}.$$

The isomorphism is then given by

$$T_{k,l} = \mathbb{1}_{\mathcal{D}_p m(p^k,p^l)\mathcal{D}_p} \cdot \operatorname{id}_{S_{L_p}^{N(\mathbb{Z}_p)}} \mapsto \chi_{D_p} T_{k,l}, \tag{5.13}$$

where have extended χ_{D_p} trivially to a quasi-character on the whole group $\mathcal{M}_p$ (see also [Ho]). □

We now define the before mentioned Satake map to further clarify the structure of $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ and to connect it to the algebra $\mathcal{H}(\mathcal{M}_p//\mathcal{D}_p)$.

$$\begin{aligned} &\mathcal{S} : \mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p) \to \mathcal{H}(\mathcal{M}_p//\mathcal{D}_p, {\omega_p}_{|S_{L_p}^{N(\mathbb{Z}_p)}}), \\ &T \mapsto \left(m \mapsto \delta(m)^{1/2} \sum_{n \in N(\mathbb{Q}_p)/N(\mathbb{Z}_p)} T(mn)_{|S_{L_p}^{N(\mathbb{Z}_p)}} \right). \end{aligned} \tag{5.14}$$

Remark 5.10. i) Note that this definition is analogous to the one given by Herzig ([He]) over a field in characteristic p. The modulus character $\delta \left(\begin{smallmatrix} m_1 & 0 \\ 0 & m_2 \end{smallmatrix} \right) = \left| \frac{m_1}{m_2} \right|_p$ is also part of the classical Satake map (see e. g. [De], Chap. 8), where it ensures that the image of the Satake map is invariant under the natural action of the Weyl group. Herzig omitted the modulus character in his definition of the Satake map as it does not produce the invariance under the action of the Weyl group. Nevertheless, we keep it in the definition of $\mathcal{S}$ since it indeed does share the property of invariance under the Weyl group.

ii) With the same arguments as after the statement of Theorem 1.2 and as in its proof (Step 0) in [He], it can be proved that $\mathcal{S}$ is a well defined map. To prove that $\mathcal{S}$ is a $\mathbb{C}-$algebra homomorphism all calculations of Step 2 in the proof of Theorem 1.2 in [He] remain valid in our situation.

As is shown in Lemma 5.6 and Corollary 5.8, the space of maps in $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ with support equal to $\mathcal{K}_p m(p^k,p^k)\mathcal{K}_p$ is two-dimensional or of higher dimension if ω_p is considered on the whole space S_{L_p}. If we restrict ourselves to an irreducible subspace of S_{L_p}, the before mentioned space is one-dimensional. On the one hand, this condition would guarantee that the Satake map (5.14) is indeed an isomorphism (without it, (5.14) is not even injective, as is easily checked). On the other hand, it is to restrictive for our purposes. So, in order to obtain an isomorphism between $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ and a subalgebra of $\mathcal{H}(\mathcal{M}_p//\mathcal{D}_p,\omega_{p_{|S_{L_p}^{N(\mathbb{Z}_p)}}})$ via (5.14), we restrict $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ to a subalgebra where the space of maps supported on the double coset $\mathcal{K}_p m(p^k,p^k)\mathcal{K}_p$ is replaced with the subspace generated by the operator

$$T_k(k_1 m(p^k,p^l)k_2) = \omega_p(k_1)\circ T(k)\circ\omega_p(k_2) \text{ with}$$

$$T(k) = \begin{cases} T^{\chi_1}(k,k) + \sum_{\substack{\chi\in\widehat{U} \text{ primitiv} \\ \chi^2\neq 1}} T^{\chi}(k,k) + T^+(k,k) + T^-(k,k), & \text{for } \mathcal{A}_p^t\oplus\mathcal{A}_p^1 \\ T^+(k,k)+T^-(k,k), & \text{for } \mathcal{A}_p^t \end{cases} \tag{5.15}$$

$$= \frac{g(D_p)}{g_{p^k}(D_p)}\,\mathrm{id}_{S_{L_p}}.$$

It will turn out that $T(k)$ is compatible with the Hecke operator $T(m(p^{-k},p^{-k}))$, see Theorem 6.9, which is the rationale for this choice.

For the next theorem we fix some notation:
Let $N(\mathcal{M}_p)$ be the normalizer of $\mathcal{M}_p$ in $\mathcal{Q}_p$. Then the group $N(\mathcal{M}_p)/\mathcal{M}_p$ is called *Weyl group*. It is isomorphic to the symmetric group S_2 and acts on $\mathcal{M}_p$ by changing the entries t_1,t_2 of a matrix $m(t_1,t_2)$.

Let $k,l\in\mathbb{Z}$ with $k\le l$. By $\mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ we mean the subalgebra of $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ generated by T with

$$T = \begin{cases} T_{k,l}, & k<l, \\ T_k, & k=l \end{cases}$$

as specified in Corollary 5.8. In order to state results for all generators of $\mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ in subsequent sections we often write $T_{k,k}$ instead of T_k. Furthermore, let

$$\tau_k = \mathbb{1}_{\mathcal{D}_p m(p^k,p^k)\mathcal{D}_p}\cdot\mathrm{id}_{S_{L_p}},$$
$$\tau_{k,l} = \mathbb{1}_{\mathcal{D}_p m(p^k,p^l)\mathcal{D}_p} + \mathbb{1}_{\mathcal{D}_p m(p^l,p^k)\mathcal{D}_p}\cdot\mathrm{id}_{S_{L_p}}.$$

Then we denote by $\mathcal{H}^W(\mathcal{M}_p//\mathcal{D}_p,\omega_{p_{|S_{L_p}^{N(\mathbb{Z}_p)}}})$ the subalgebra of $\mathcal{H}(\mathcal{M}_p//\mathcal{D}_p,\omega_{p_{|S_{L_p}^{N(\mathbb{Z}_p)}}})$ generated by $\tau_{k,l}$ and τ_k, which is nothing else but subalgebra of all elements of $\mathcal{H}(\mathcal{M}_p//\mathcal{D}_p,\omega_{p_{|S_{L_p}^{N(\mathbb{Z}_p)}}})$ invariant under the Weyl group W.

Theorem 5.11. *Let p be a prime dividing $|D|$ and L_p'/L_p anisotropic.*
Then the Hecke algebras $\mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ and $\mathcal{H}^W(\mathcal{M}_p//\mathcal{D}_p,\omega_{p_{|S_{L_p}^{N(\mathbb{Z}_p)}}})$ are isomorphic.

Proof. In view of Remark 5.10, it suffices to prove that $\mathcal{S}$ is injective and surjective. To this end, we compute $\mathcal{S}(T)$ for a non-zero $T\in\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$. By Corollary 5.8, we may assume that T is either $T_{k,l}$ or T_k.

We first consider the case $k<l$. Thus, $T=T_{k,l}\in\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ with $\mathrm{supp}(T_{k,l}) = \mathcal{K}_p m(p^k,p^l)\mathcal{K}_p$. Let $m(p^i,p^j)\in\mathcal{M}_p$ for arbitrary $i,j\in\mathbb{Z}$ with $i\le j$ and $i+j$ a square. One can prove (see [De], Lemma 8.24) that $m(p^i,p^j)N(\mathbb{Q}_p)\cap\mathcal{K}_p m(p^k,p^l)\mathcal{K}_p\neq\emptyset$ if and only if $i,j\ge k$ and $i+j=k+l$. Therefore,

$$\mathrm{supp}(\mathcal{S}T_{k,l})\subset\{\mathcal{D}_p m(p^\nu,p^{k+l-\nu})\mathcal{D}_p \mid \nu=k,\dots,l\}. \tag{5.16}$$

For $x = 0$ it is obvious that $m(p^\nu, p^{k+l-\nu})n(0)$ lies in $\mathcal{K}_p m(p^k, p^l)\mathcal{K}_p$ if and only $\nu = k$ or $\nu = l$. Let $0 \neq x = p^r s$ with $r < 0$ and $s \in \mathbb{Z}_p^\times$. Employing the Cartan decomposition produces for all ν

$$m(p^\nu, p^{k+l-\nu})n(p^r s) = n_-(p^{k+l-2\nu-r}s^{-1})m(p^{\nu+r}, p^{k+l-\nu-r})m(s)n(-p^{-r}s^{-1})w \tag{5.17}$$

if $k+l-2\nu-r \geq 0$. It follows for $\nu = k, \dots, l$ that the matrix $m(p^\nu, p^{k+l-\nu})n(p^r s)$ lies in the double coset $\mathcal{K}_p m(p^k, p^l)\mathcal{K}_p$ if and only if $r = k - \nu$. Thus, for $\nu = k+1, \dots, l-1$ the sum $\sum_{x \in \mathbb{Q}_p/\mathbb{Z}_p} T_{k,l}(m(p^\nu, p^{k+l-\nu})n(x))$ runs over all elements of the form $x = p^{k-\nu}s$, s traversing the set

$$\mathcal{U}(\nu) = \left\{ \sum_{i=0}^{\nu-k-1} x_i p^i \mid x_0 \in (\mathbb{Z}/p\mathbb{Z})^\times \text{ and } x_i \in \mathbb{Z}/p\mathbb{Z},\ i = 1, \dots, \nu-k-1 \right\}.$$

If $k+l-2\nu-r < 0$, we find

$$m(p^\nu, p^{k+l-\nu})n(p^r s) = n(p^{2\nu+r-k-l}s)m(p^\nu, p^{k+l-\nu}). \tag{5.18}$$

Obviously, $m(p^\nu, p^{k+l-\nu})n(p^r s)$ is contained in $\mathcal{K}_p m(p^k, p^l)\mathcal{K}_p$ if and only if $\nu = k$ or $\nu = l$. In the latter case,

$$m(p^\nu, p^{k+l-\nu})n(p^r s) = n(p^{l-k+r}s)w^{-1}m(p^k, p^l)w. \tag{5.19}$$

As $k+l-2\nu-r < 0$ is equivalent to $r > k+l-2\nu$, for $\nu = l$, the sum $\sum_{x \in \mathbb{Q}_p/\mathbb{Z}_p} T_{k,l}(m(p^l, p^k)n(x))$ runs over all $x \in \mathbb{Q}_p/\mathbb{Z}_p$ with $|x|_p < l-k$. Assuming a representation of the form $x = p^r s$, we may put $r = k-l$ and write

$$\sum_{x \in \mathbb{Q}_p/\mathbb{Z}_p} T_{k,l}(m(p^l, p^k)n(x)) = \sum_{s \in \mathcal{U}(l)^0} T_{k,l}(m(p^l, p^k)n(p^{k-l}s)),$$

where

$$\mathcal{U}(l)^0 = \left\{ \sum_{i=0}^{l-k-1} x_i p^i \mid x_0 = 0 \text{ and } x_i \in \mathbb{Z}/p\mathbb{Z},\ i = 1, \dots, l-k-1 \right\}.$$

Note that $|s|_p < 1$ for all $s \in \mathcal{U}(l)^0$.

Consequently, $\mathcal{U}(l) \cup \mathcal{U}(l)^0$ contains all principal parts x in $\mathbb{Q}_p/\mathbb{Z}_p$ with $\nu_p(x) \geq k-l$. As already pointed out, these are all x, for which $T_{k,l}(m(p^l, p^k)n(x))$ is non-zero.

By means of the decomposition (5.17), we are now able to compute $(\mathcal{S}T_{k,l})(m(p^\nu, p^{k+l-\nu}))$ explicitly for any $\nu \in \{k, \dots, l\}$. Since the computations for $\nu = l$ are more complicated, we treat them separately afterwards. Thus, let $\nu \in \{k, \dots, l-1\}$. Then

(5.20)

$$\begin{aligned}
&\sum_{x \in \mathbb{Q}_p/\mathbb{Z}_p} T_{k,l}\Big(m(p^\nu, p^{k+l-\nu})n(x)\Big)_{|S_{L_p}^{N(\mathbb{Z}_p)}} \\
&= \begin{cases} \sum_{s \in \mathcal{U}(\nu)} T_{k,l}\left(m(p^\nu, p^{k+l-\nu})n(p^{k-\nu}s)\right)_{|S_{L_p}^{N(\mathbb{Z}_p)}}, & \nu \neq k, \\ T_{k,l}(m(p^k, p^l))_{|S_{L_p}^{N(\mathbb{Z}_p)}}, & \nu = k \end{cases} \\
&= \begin{cases} \sum_{s \in \mathcal{U}(\nu)} \omega_p(n_-(p^{l-\nu}s^{-1})) \circ T_{k,l}(m(p^k, p^l)) \circ \omega_p(m(s)n(-p^{\nu-k}s^{-1})w)_{|S_{L_p}^{N(\mathbb{Z}_p)}}, & \nu \neq k, \\ T_{k,l}(m(p^k, p^l))_{|S_{L_p}^{N(\mathbb{Z}_p)}}, & \nu = k. \end{cases}
\end{aligned}$$

Since the level of L_p'/L_p is p, the last expression in (5.20) for $\nu \neq k, l$ simplifies to

$$\sum_{s\in U(\nu)} T_{k,l}(m(p^k,p^l)) \circ \omega_p\left(m(s)w\right)_{|S_{L_p}^{N(\mathbb{Z}_p)}}$$

With the help of the explicit formulas (4.7) of ω_p and Lemma 5.6, we obtain

$$\begin{aligned}
&(\mathcal{S}T_{k,l})(m(p^\nu,p^{k+l-\nu}))\varphi_p^{(0)}\\
&= \delta(m(p^\nu,p^{k+l-\nu}))^{1/2}\frac{\gamma_p(L_p'/L_p)}{|L_p'/L_p|^{1/2}}\sum_{\gamma\in L_p'/L_p}\sum_{s\in\mathcal{U}(\nu)}\left(\frac{s}{|L_p'/L_p|}\right)T_{k,l}(m(p^k,p^l))\varphi_p^{(s^{-1}\gamma)}\\
&= \begin{cases} 0, & \text{if } |L_p'/L_p| = p,\\ \delta(m(p^\nu,p^{k+l-\nu}))^{1/2}\gamma_p(L_p'/L_p)|L_p'/L_p|^{1/2}|\mathcal{U}(\nu)|\varphi_p^{(0)}, & \text{if } |L_p'/L_p| = p^2.\end{cases}
\end{aligned}$$

In view of the discussion above, for $\nu = l$ we have

(5.21)
$$\begin{aligned}
&\sum_{x\in\mathbb{Q}_p/\mathbb{Z}_p} T_{k,l}(m(p^l,p^k)n(x))\varphi_p^{(0)} =\\
&\sum_{s\in\mathcal{U}(l)}\omega_p(n_-(s^{-1}))T(m(p^k,p^l))\omega_p(m(s)w)\varphi_p^{(0)} + \sum_{s\in\mathcal{U}(l)^0}\omega_p(w^{-1})T_{k,l}(m(p^k,p^l))\omega_p(w)\varphi_p^{(0)}.
\end{aligned}$$

With the help of Lemma 4.1 and the calculations before, it can be verified that the first summand of the above expression is equal to

$$\begin{aligned}
&\left(\frac{-1}{|L_p'/L_p|}\right)\gamma_p(L_p'/L_p)^2\sum_{s\in\mathcal{U}(l)}\sum_{\nu_p\in L_p'/L_p}\psi_p(sq(\nu_p))\varphi_p^{(\nu_p)}\\
&= \sum_{\nu_p\in L_p'/L_p}\left(|\mathcal{U}(l)^0|\sum_{x_0\in(\mathbb{Z}/p\mathbb{Z})^\times} e(x_0q(\nu_p))\right)\varphi_p^{(\nu_p)},
\end{aligned} \tag{5.22}$$

where we exploited for the last equation the fact that level of L_p is p and that $\gamma_p(L_p'/L_p)^2 = \left(\frac{-1}{|L_p'/L_p|}\right)$ (see e. g. [Ze], p. 73). Similarly, the second summand can be evaluated to be

$$|\mathcal{U}(l)^0|\sum_{\nu_p\in L_p'/L_p}\varphi_p^{(\nu_p)}. \tag{5.23}$$

Replacing the right-hand side of (5.21) with (5.22) and (5.23), yields

$$\begin{aligned}
&\sum_{x\in\mathbb{Q}_p/\mathbb{Z}_p} T_{k,l}(m(p^l,p^k)n(x))\varphi_p^{(0)}\\
&= |\mathcal{U}(l)^0|\sum_{\nu_p\in L_p'/L_p}\left(\sum_{x\in\mathbb{Z}/p\mathbb{Z}} e(xq(\nu_p))\right)\varphi_p^{(\nu_p)}\\
&= |\mathcal{U}(l)^0|p\varphi_p^{(0)}.
\end{aligned}$$

Here we have used the standard formula for the Gauss sum $\sum_{x \in \mathbb{Z}/p\mathbb{Z}} e(xq(\nu_p))$. This leads us finally to

(5.24)

$$(\mathcal{S}T_{k,l})(m(t_1,t_2)) = \frac{g(D_p)}{g_{p^l}(D_p)}\delta(m(t_1,t_2))^{1/2}\times$$

$$\begin{cases} \mathbb{1}_{\mathcal{D}_p m(p^k,p^l)\mathcal{D}_p} \operatorname{id}_{S_{L_p}^{N(\mathbb{Z}_p)}}, & m(t_1,t_2) = m(p^k,p^l), \\ \delta_p \gamma_p(L_p'/L_p)|L_p'/L_p|^{1/2}|\mathcal{U}(\nu)|\mathbb{1}_{\mathcal{D}_p m(p^\nu,p^{k+l-\nu})\mathcal{D}_p} \operatorname{id}_{S_{L_p}^{N(\mathbb{Z}_p)}}, & m(t_1,t_2) = m(p^\nu,p^{k+l-\nu}), \nu \neq k,l \\ p^{l-k}\mathbb{1}_{\mathcal{D}_p m(p^l,p^k)\mathcal{D}_p} \operatorname{id}_{S_{L_p}^{N(\mathbb{Z}_p)}}, & m(t_1,t_2) = m(p^l,p^k), \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$= \frac{g(D_p)}{g_{p^l}(D_p)} p^{\frac{1}{2}(l-k)}\times$$

$$\begin{cases} \mathbb{1}_{\mathcal{D}_p m(p^k,p^l)\mathcal{D}_p} \operatorname{id}_{S_{L_p}^{N(\mathbb{Z}_p)}}, & m(t_1,t_2) = m(p^k,p^l), \\ \delta_p \gamma_p(L_p'/L_p)\mathbb{1}_{\mathcal{D}_p m(p^\nu,p^{k+l-\nu})\mathcal{D}_p} \operatorname{id}_{S_{L_p}^{N(\mathbb{Z}_p)}}, & m(t_1,t_2) = m(p^\nu,p^{k+l-\nu}), \nu \neq k,l \\ \mathbb{1}_{\mathcal{D}_p m(p^l,p^k)\mathcal{D}_p} \operatorname{id}_{S_{L_p}^{N(\mathbb{Z}_p)}}, & m(t_1,t_2) = m(p^l,p^k), \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

where $\delta_p = 1$ if $|L_p'/L_p| = p^2$ and zero otherwise.

If $k = l$, it follows from (5.16) that $\operatorname{supp}(\mathcal{S}T_k) = \mathcal{D}_p m(p^k,p^k)\mathcal{D}_p$. The same thoughts as for $k < l$ after equation (5.19) yield

$$\begin{aligned}(\mathcal{S}T_k(m(p^k,p^k)) &= T_k(m(p^k,p^k))_{|S_{L_p}^{N(\mathbb{Z}_p)}} \\ &= \frac{g(D_p)}{g_{p^k}(D_p)}\mathbb{1}_{\mathcal{D}_p m(p^k,p^k)\mathcal{D}_p} \operatorname{id}_{S_{L_p}^{N(\mathbb{Z}_p)}}.\end{aligned}$$

From the above follows immediately that $\mathcal{S}$ is injective. For the surjectivity it suffices to proof that τ_k and $\tau_{k,l}$ are contained in the image of $\mathcal{S}$. This can be done almost verbatim as in [De], p. 212. □

Whenever p *divides* $|D|$ and we deal with Elements $T_{k,l}$, T_k or $\tau_{k,l}$, τ_k of either of the Hecke algebras $\mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ or $\mathcal{H}^W(\mathcal{M}_p//\mathcal{D}_p,\omega_{p_{|S_{L_p}^{N(\mathbb{Z}_p)}}})$, we mean the above stated and assume that L_p'/L_p is *anisotropic.*

5.2. **The case of primes p not dividing $|D|$.** We denote with $\mathcal{H}(\mathcal{M}_p//\mathcal{D}_p)^W$ the subalgebra of all elements of $\mathcal{H}(\mathcal{M}_p//\mathcal{D}_p)$ invariant under the Weyl group W. In this case it easily seen that $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ is isomorphic to $\mathcal{H}(\mathcal{M}_p//\mathcal{D}_p)^W$.

Theorem 5.12. *Let p be a prime coprime to $|D|$. Then the Hecke algebras $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ and $\mathcal{H}(\mathcal{M}_p//\mathcal{D}_p)^W$ are isomorphic as algebras.*

Proof. By Lemma 3.4 in [St], we know that L_p is unimodular and ω_p is the trivial representation on the space $S_{L_p} = \mathbb{C}\varphi_p^{(0)}$. A basis of $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ is then given by

$$\left\{\mathbb{1}_{\mathcal{K}_p m(p^k,p^l)\mathcal{K}_p}\cdot \operatorname{id}_{S_{L_p}} \mid k,l \in \Lambda_+\right\},$$

where $\mathbb{1}_{\mathcal{K}_p m(p^k,p^l)\mathcal{K}_p}$ is the characteristic function of $\mathcal{K}_p m(p^k,p^l)\mathcal{K}_p$. The composition of

$$F : \mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p) \to \mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p), \quad \mathbb{1}_{\mathcal{K}_p m(p^k,p^l)\mathcal{K}_p} \cdot \mathrm{id}_{S_{L_p}} \mapsto \mathbb{1}_{\mathcal{K}_p m(p^k,p^l)\mathcal{K}_p}$$

with the classical Satake map (see e. g. [De] or [Ca]) gives the desired isomorphism. □

For later purposes, we conclude this chapter with the observation that $\mathcal{H}(\mathcal{M}_p//\mathcal{D}_p)$ can be identified with the group algebra $\mathbb{C}[\mathcal{M}_p/\mathcal{D}_p]$ via

$$\mathbb{1}_{\mathcal{D}_p m(p^k,p^l)\mathcal{D}_p} \mapsto \mathbb{1}_{m(p^k,p^l)\mathcal{D}_p}.$$

6. Hecke operators and their relation to $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$

In his thesis [We], Werner introduced a generalized version of the Hecke operators $T(m^2)^*$ defined in [BS] (see Section 3). Most notably, an extension of the Weil representation from $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ to $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ (N is the level of L) was specified without the condition that the parameter m has to be a square. Werner also laid the foundation of an adelic description of these generalized Hecke operators. In particular, he assigned to each vector valued modular form a vector valued automorphic form on $\mathrm{GL}_2(\mathbb{A})$. In this section we continue this work and embed it into a more general framework of vector valued automorphic forms. However, we stick to the Hecke operators given in [BS]. As a consequence, we have to work with the extension of the Weil representation as stated in Section 3 and its adelic counterpart in Section 4.

6.1. **Relation between vector valued automorphic forms and vector valued modular forms.** Instead of working with $\mathrm{GL}_2(\mathbb{A})$, we consider the restricted product

$$\mathcal{G}(\mathbb{A}) = \prod_{p\le\infty}{}' \mathcal{Q}_p = \left\{ (g_p) \in \prod_{p\le\infty} \mathcal{Q}_p \mid g_p \in \mathcal{K}_p \text{ for almost all primes } p \right\}, \tag{6.1}$$

where

$$\mathcal{Q}_\infty = \{ M \in \mathrm{GL}_2(\mathbb{R}) \mid \det(M) \in (\mathbb{R}^\times)^2 \}.$$

Note that $\mathcal{K}_\infty = \mathrm{SO}(2)$ is a subgroup of $\mathcal{Q}_\infty$. The group $\mathcal{G}(\mathbb{Q})$ can be embedded diagonally as a discrete subgroup of $\mathcal{G}(\mathbb{A})$. An important decomposition for $\mathrm{GL}_2(\mathbb{A})$, which will be needed for the definition of automorphic forms, is the strong approximation. An analogous result holds for $\mathcal{G}(\mathbb{A})$.

Theorem 6.1. *Let $\mathcal{K} = \prod_{p<\infty} \mathcal{K}_p \subset \mathrm{GL}_2(\mathbb{A}_f)$. Then*

$$\mathcal{G}(\mathbb{A}) = \mathcal{G}(\mathbb{Q})(\mathcal{Q}_\infty \times \mathcal{K}). \tag{6.2}$$

More generally, let $\mathcal{U} = \prod_{p<\infty} \mathcal{U}_p$ be any open compact subgroup of $\mathcal{K}$ with the property that $\det(\mathcal{U}) = (\widehat{\mathbb{Z}}^\times)^2$. Then

$$\begin{aligned} &\mathcal{G}(\mathbb{A}_f) = \mathcal{G}(\mathbb{Q}) \cdot \mathcal{U} \text{ and} \\ &\mathcal{G}(\mathbb{A}) = \mathcal{G}(\mathbb{Q})(\mathcal{Q}_\infty \times \mathcal{U}). \end{aligned} \tag{6.3}$$

Proof. A proof for the classical result for $\mathrm{GL}_2(\mathbb{A})$ can be found in many places among them in [KL], Section 5.2 and Section 6.3. One can check that the the proofs of Proposition 5.10, Proposition 6.5 and Theorem 6.8 of [KL] carry over to the analogous statements in our setting. □

In [KL] and [Ge] functions $f : \mathrm{GL}_2(\mathbb{Q}) \setminus \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ with certain properties were related to (scalar valued) modular forms. Here we consider $\mathcal{G}(\mathbb{Q})$-invariant and S_L-valued functions

$$F : \mathcal{G}(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{A}) \to S_L$$

with a similar goal. With respect to the basis $(\varphi_\mu)_{\mu\in D}$ of S_L such a function can be written in the form $F = \sum_{\mu\in D} F_\mu \varphi_\mu$. In view of (4.5) and (4.6), we will consider only *factorizable functions*, that is, only those S_L-valued functions F, which possess a decompostion of the form

$$F(\gamma(g_\infty \times g_f)) = \bigotimes_{p<\infty} F_p(g_\infty, g_p),$$

where

$$F_p(g_\infty, g_p) = \begin{cases} \sum_{\lambda\in L_p'/L_p} F_{\lambda,\infty}(g_\infty) F_{\lambda,p}(g_p)\varphi_p^{(\lambda)}, & p \mid |D|, \\ \varphi_p^{(0)}, & p \nmid |D|. \end{cases}$$

Using the bilinearity of the tensor product, we have

$$F(g) = \sum_{(\lambda_p)_p \in \bigoplus_{p<\infty} L_p'/L_p} F_{\lambda_p\infty}(g_\infty) \prod_{p<\infty} F_{\lambda_p,p}(g_p) \bigotimes_{p<\infty} \varphi_p^{(\lambda_p)}.$$

Note that F is well defined since any occurring sum, product or tensor product is finite.

We denote the subspace of all these functions $F : \mathcal{G}(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{A}) \to S_L$ with $\mathcal{F}_L$. Associated to the Weil representation ω_f of $\mathcal{G}(\mathbb{A}_f)$ on the space S_L and analogous to the corresponding scalar valued space in [KL], we define

(6.4)

$$L^2(\mathcal{G}(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{A}), \omega_f) = \left\{ F \in \mathcal{F}_L \;\middle|\; \begin{array}{ll} \text{i)} & F_\mu \text{ is measurable for all } \mu \in D \\ \text{ii)} & F(zg) = \omega_f(z_f)^{-1}F(g) \text{ for all} \\ & z = z_{\mathbb{Q}}(z_\infty \times z_f) \in \mathcal{Z}(\mathbb{A}) \\ \text{iii)} & \int_{\overline{\mathcal{G}}(\mathbb{Q})\setminus\overline{\mathcal{G}}(\mathbb{A})} \|F(g)\|^2 dg < \infty \end{array} \right\}$$

and

$$L_0^2(\omega_f) = \left\{ F \in L^2(\mathcal{G}(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{A}), \omega_f) \;\middle|\; \int_{N(\mathbb{Q})\setminus N(\mathbb{A})} F_\mu(ng)dn = 0 \text{ for all } \mu \in D, \text{ a. e. } g \in \mathcal{G}(\mathbb{A}) \right\}.$$

Here by

i) $\|F(g)\|^2$ we mean $\langle F(g), F(g)\rangle$ as defined in (4.4),
ii) $\overline{\mathcal{G}}(R) = \mathcal{Z}(R) \setminus \mathcal{G}(R)$, where $\mathcal{Z}(R)$ is the center of $\mathcal{G}(R)$,
iii) dg and dn we mean the Haar measure on $\overline{\mathcal{G}}(\mathbb{Q}) \setminus \overline{\mathcal{G}}(\mathbb{A})$ and $N(\mathbb{Q}) \setminus N(\mathbb{A})$, respectively.

Measurability for each component function F_μ is meant in the sense of Proposition 7.15 of [KL]: F_μ can be written as a product $\prod_{p\le\infty} F_{\mu,p}(g_p)$, each component satisfying:

i) $F_{\mu,p} : \mathcal{Q}_p \to \mathbb{C}$ is measurable for all $p \le \infty$
ii) $F_{\mu,p_{|\mathcal{K}_p}} = 1$ for all $p \notin S$, where S is finite set of places.

The above integrals over $\overline{\mathcal{G}}(\mathbb{Q}) \setminus \overline{\mathcal{G}}(\mathbb{A})$ and $N(\mathbb{Q}) \setminus N(\mathbb{A})$ are explained in [KL], Proposition 7.43 and Proposition 12.2, and meant in the very same way. Also note that the integral in iii) of $L^2(\mathcal{G}(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{A}), \omega_f)$ is well defined as F satisfies ii) and the Weil representation ω_f is unitary with respect to $\langle\cdot,\cdot\rangle$.

Werner assigned in [We], Def. 49, a $\mathbb{C}[D]$-valued Function F_f on $\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})$ to a cusp form $f \in S_\kappa(\rho_L)$. We adopt his definition to our setting, which basically means that we replace the group ring with the isomorphic space S_L.

Definition 6.2. Let $f \in S_\kappa(\rho_L)$ and $g \in \mathcal{G}(\mathbb{A})$ with $g = \gamma(g_\infty \times k)$, where $\gamma \in \mathcal{G}(\mathbb{Q}), g_\infty \in \mathcal{Q}_\infty$ and $k \in \mathcal{K}$. Then in terms of this decomposition we define a map $\mathscr{A}$

$$f \mapsto \mathscr{A}(f) = F_f \quad \text{with } F_f(g) = \omega_f(k)^{-1} j(g_\infty, i)^{-\kappa} f(g_\infty i). \tag{6.5}$$

Lemma 50 in [We] shows that the definition of F_f in (6.5) is independent from the decomposition of g. Moreover, from its definition it follows immediately that F_f is $\mathcal{G}(\mathbb{Q})$-invariant.

Proposition 6.3. *Let $f \in S_\kappa(\rho_L)$. Then the assigned function F_f on $\mathcal{G}(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{A})$ lies in the space $L^2(\mathcal{G}(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{A}), \omega_f)$.*

Proof. i) By definition the μ-th component of F_f is given by

$$\begin{aligned}(F_f)_\mu(g) &= \langle \omega_f(k)^{-1} j(g_\infty, i)^{-\kappa} f(g_\infty i), \varphi^{(\mu)}\rangle \\ &= j(g_\infty, i)^{-\kappa} \sum_{\lambda \in D} f_\lambda(g_\infty i) \prod_{p<\infty} \langle \omega_p^{-1}(k_p)\varphi_p^{(\lambda_p)}, \varphi_p^{(\mu_p)}\rangle,\end{aligned} \tag{6.6}$$

where $g = \gamma(g_\infty \times k)$. It is well known that $j(g_\infty, i)^{-\kappa} f_\lambda(g_\infty i)$ is measurable on $\mathcal{Q}_\infty$ as f_λ is a scalar valued cusp form for $\Gamma(N)$ (cf. [Ge], §2, for this case). As a result of the discussion in Chapter 4, we have that ω_p is trivial for all $p \nmid N$. For $p \mid N$ we find by means of the explicit formulas of ω_p (see e. g. [BY], p. 645, or [St], Lemma 3.4) that ω_p is trivial on the subgroup

$$\mathcal{K}_p(p^{\mathrm{ord}_p(D)}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{K}_p \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod p^{\mathrm{ord}_p(D)}\mathbb{Z}_p \right\}$$

and factors thereby through $\mathcal{K}_p/\mathcal{K}_p(p^{\mathrm{ord}_p(D)})$ for each p dividing N. Since $\mathcal{K}_p(p^{\mathrm{ord}_p(D)})$ has as compact subgroup a finite measure, $\langle \omega_p^{-1}(k_p)\varphi_p^{\mu_p}, \varphi_p^{\mu_p}\rangle$ is a measurable function for all primes p. We then obtain that $(F_f)_\mu$ is measurable in the above stated sense.

ii) Let $z = z_\mathbb{Q}(z_\infty \times z_f) \in \mathcal{Z}(\mathbb{A})$. Then it follows immediately from the definition of F_f that $F_f(zg) = \omega_f(z_f)^{-1} F_f(g)$.

iii) It can be verified that Proposition 7.43 and the discussion before of [KL] is also valid in our situation. We have to check that all steps of the proof are still working if we replace the involved groups by the corresponding groups in our setting. This is in fact the case, some steps are even easier since we only have to deal with matrices whose determinant is a square. As a result, we may replace the integral over $\overline{\mathcal{G}}(\mathbb{Q}) \setminus \overline{\mathcal{G}}(\mathbb{A})$ with the corresponding integral over $D\mathcal{K}_\infty \times \mathcal{K}$. Here D is a fundamental domain for $\Gamma(1) \setminus \mathbb{H}$ interpreted as subset of $\mathrm{SL}_2(\mathbb{R})$. Following the proof of Proposition 12.15 in [KL], we find for F_f

$$\begin{aligned}\int_{\overline{\mathcal{G}}(\mathbb{Q})\setminus\overline{\mathcal{G}}(\mathbb{A})} \|F_f(g)\|^2 dg &= \int_{D\mathcal{K}_\infty} \int_{\mathcal{K}} \|F_f(g \times k)\|^2 dk dg \\ &= \int_D \|j(g_\infty, i)^{-\kappa} f(g_\infty i)\|^2 dg,\end{aligned} \tag{6.7}$$

where we have used that ω_f is unitary with respect to $\langle\cdot,\cdot\rangle$ and that the Haar measure on $\mathcal{Q}_p$ is normalized to be equal to one on $\mathcal{K}_p$ for $p \le \infty$. If we identify $g_\infty i$ with an element $\tau \in \Gamma(1) \setminus \mathbb{H}$, the last integral in (6.7) becomes

$$\int_{\Gamma(1)\setminus\mathbb{H}} \|f(\tau)\|^2 \operatorname{Im}(\tau)^\kappa \frac{dxdy}{y^2} < \infty,$$

which is the Petersson norm of $f \in S_\kappa(\rho_L)$. Therefore, the L^2-norm of F_f is finite. □

Lemma 6.4. *Let $f \in S_\kappa(\rho_L)$ and F_f the assigned automorphic form given by* (6.5). *Then*

$$\int_{N(\mathbb{Q})\setminus N(\mathbb{A})} F_\mu(ng)dn = 0$$

for almost every $g \in \mathcal{G}(\mathbb{A})$ and all $\mu \in D$.

Proof. The proof proceeds along the lines of the one of Proposition 12.2 in [KL]. Let $n = n(x_\mathbb{Q})(n(x_\infty) \times n(x_f)) \in N(\mathbb{A})$ and $g = \gamma(g_\infty \times g_f) \in \mathcal{G}(\mathbb{A})$. Then the definition of F_f and ω_f yields

$$\begin{aligned} F_\mu(ng) &= \langle j(g_\infty, i)^{-\kappa} j(n(x_\infty), g_\infty i)^{-\kappa} \omega_f^{-1}(g_f)\omega_f^{-1}(n(x_f)) f(n(x_\infty)(g_\infty i)), \varphi_\mu\rangle \\ &= j(g_\infty, i)^{-\kappa} j(n(x_\infty), g_\infty i)^{-\kappa} \sum_{\nu\in D} \psi_f(-x_f q(\nu)) f_\nu(n(x_\infty)(g_\infty i))\langle \omega_f^{-1}(g_f)\varphi_\nu, \varphi_\mu\rangle. \end{aligned}$$

As suggested in [KL], Prop. 12.2., we calculate more generally for $r \in \mathbb{Q}$

$$\begin{aligned} &\int_{N(\mathbb{Q})\setminus N(\mathbb{A})} F_\mu(n(x)g)\psi(rx)dx \\ &= j(g_\infty, i)^{-\kappa} \sum_{\nu\in D} \langle \omega_f^{-1}(g_f)\varphi_\nu, \varphi_\mu\rangle \times \\ &\int_{\mathbb{N}(\mathbb{Z})\setminus(N(\mathbb{R})\times N(\widehat{\mathbb{Z}}))} \psi_f(-x_f q(\nu)) f_\nu(n(x_\infty)(g_\infty i))\psi_\infty(r x_\infty)\psi_f(r x_f)dx_f dx_\infty. \end{aligned} \tag{6.8}$$

We can write the integral in the last expression as

$$\int_0^1 f_\nu(n(x_\infty)(g_\infty i))\psi_\infty(r x_\infty) \int_{N(\widehat{\mathbb{Z}})} \psi_f((r - q(\nu))x_f)dx_f dx_\infty,$$

where the integral over $N(\widehat{\mathbb{Z}})$ is one if and only if $r \in \mathbb{Z} + q(\nu)$. For such r (note that $\psi_\infty(x_\infty) = e(-x_\infty)$), taking into account that $\int_0^1 f_\nu(x_\infty+\tau)e(-r x_\infty)dx_\infty = e(r \operatorname{Re}(\tau))a(\nu, r)$, where $a(\nu, r)$ is the Fourier coefficient of f with respect to (ν, r), we finally obtain

$$\int_{N(\mathbb{Q})\setminus N(\mathbb{A})} F_\mu(n(x)g)\psi(rx)dx = j(g_\infty, i)^{-\kappa} \sum_{\nu\in D} \langle \omega_f^{-1}(g_f)\varphi_\nu, \varphi_\mu\rangle e(r \operatorname{Re}(\tau))a(\nu, r),$$

where $\tau = g_\infty i$. Since f is a cusp form, we have that for $r = 0$ all coefficients $a(\nu, r)$ vanish. This gives the desired result. □

The image of $S_\kappa(\rho_L)$ under the map $\mathscr{A}$ in (6.5) can be characterized more closely:

Theorem 6.5. *Let $A_\kappa(\omega_f)$ be the space of functions $F \in L_0^2(\omega)$ satisfying*

i) $F(gk) = \omega_f(k)^{-1}F(g)$ *for all $k \in \mathcal{K}$ and all $g \in \mathcal{G}(\mathbb{A})$*
ii) $F(g \left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta)\end{smallmatrix}\right)) = e^{i\kappa\theta}F(g)$ *for all $\theta \in [0, 2\pi)$ and all $g \in \mathcal{G}(\mathbb{A})$*

iii) *All the components F_μ of F, considered as a function of $\mathcal{Q}_\infty$ alone, satisfy the differential equation $LF_\mu = 0$. Here L is the differential operator given by*

$$L = e^{-2i\theta}\left(-2iy\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + i\frac{\partial}{\partial \theta}\right) \tag{6.9}$$

with respect to the coordinates referring to the decomposition

$$g_\infty = z_\infty \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \tag{6.10}$$

of $g_\infty \in \mathcal{Q}_\infty$.

Then the map $\mathscr{A}$ defines an isometry from $S_\kappa(\rho_L)$ onto $A_\kappa(\omega_f)$.

Proof. This theorem is well known for scalar valued automorphic forms, see e. g. [Ge] or [KL]. Most parts of its proof can be settled with reference to the proof of its scalar valued analogue.

Let $f \in S_\kappa(\rho_L)$. It follows from Proposition 6.3 and Lemma 6.4 that $F_f \in L_0^2(\omega_f)$. The assertion in i) is proved in [We], Theorem 51, the one in ii) results from a straightforward calculation analogous to the scalar valued case (see [KL], Proposition 12.5). For iii) note that $F_\mu(g_\infty \times 1_f) = y^{k/2}e^{ik\theta}f_\mu(x+iy)$ if we decompose $g_\infty \in \mathcal{Q}_\infty$ according to (6.10). The same proof as in [KL], applied to each component F_μ, establishes the result using the assumption $f \in S_\kappa(\rho_L)$.

Kudla [Ku] defined a map that assigns to a vector valued function F on $\mathcal{G}(\mathbb{A})$ a vector valued function f_F on $\mathbb{H}$:

$$F \mapsto f_F,\ f_F(\tau) = j(g_\tau, i)^\kappa F(g_\tau \times 1_f), \tag{6.11}$$

where $g_\tau = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}$ and $\tau = g_\tau i = x + iy \in \mathbb{H}$. It is easily seen that this map is well-defined and that it is the inverse map of $\mathscr{A}$ (see [KL], Prop. 12.5, for the corresponding scalar valued result). It remains to show that f_F is an element of $S_\kappa(\rho_L)$ for any $F \in A_\kappa(\omega_f)$. Kudla proved that f_F transforms like a vector valued modular form with respect to ω_f if $F \in A_\kappa(\omega_f)$ ([Ku], Lemma 1.1). Since each component of F satisfies the differential equation in iii), it follows that each component of f_F is holomorphic on the upper half plane (see [KL], Prop. 12.5). In view of these two properties, f_F possess a Fourier expansion, see [Br1], p. 18. By Proposition 6.3 we know that the Petersson norm of f_F coincides with the L^2 norm of F, it is in particular finite. One can prove in the same way as in Prop. 3.39 of [KL] that f_F is an element of $S_\kappa(\rho_L)$. Thus, the map in (6.5) is surjective and an isometry. □

6.2. **The action of $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ on $A_\kappa(\omega_f)$.** The goal of this subsection is to define an action of $\mathcal{G}(\mathbb{A})$ via the Hecke algebra $\mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ on the space $A_\kappa(\omega_f)$ of vector valued automorphic forms. Since $\mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ acts only on the p-component of an element $F \in A_\kappa(\omega_f)$, we need to complement the contribution of $\mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ with suitable operators on the other places. The envisaged action will be defined in such a way that it is compatible with the action of Hecke operators on $S_\kappa(\rho_L)$. Werner proposed in [We], Chapter 6, the definition of an adelic vector valued Hecke operator mimicking Gelbart's approach of an adelic scalar valued Hecke operator. Our approach is slightly more general and transfers the action of the spherical Hecke algebra in [Mu1], § 6, to the vector valued setting.

Definition 6.6. Let $p \in \mathbb{Z}$ be a fixed prime, $g = \gamma(g_\infty \times g_f) \in \mathcal{G}(\mathbb{A})$ and $T_p \in \mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$. Then we define for a fixed $h \in \mathcal{G}(\mathbb{A})$

$$R^{T_p}(h) : \mathcal{F}_L \to \mathcal{F}_L, \quad F \mapsto R^{T_p}(h)F = \bigotimes_{q<\infty} R_q^{T_p}(h_q)F_q \tag{6.12}$$

with

$$R_q^{T_p}(h_q)F_q(g_q) = \begin{cases} F_q(g_\infty, g_q h_q), & q \neq p \\ T_p(h_p)(F_p(g_\infty, g_p)), & q = p. \end{cases} \tag{6.13}$$

The operator

$$\mathcal{T}^{T_p} : A_\kappa(\omega_f) \to A_\kappa(\omega_f), \quad \mathcal{T}^{T_p}(F)(g) = \sum_{x_p \in \mathcal{Q}_p/\mathcal{K}_p} R^{T_p}(\iota_p(x_p))F(g\iota_p(x_p)) \tag{6.14}$$

can be interpreted as a vector valued analogue of the construction in [Mu1] (see Section 2 for the definition of ι_p). For the sake of better readability, we omit the argument g_∞ in the subsequent calculations and assume tacitly that the local functions also depend on g_∞.

Remark 6.7. i) If we decompose $\mathcal{T}^{T_p}$ into into its components, we obtain

$$\mathcal{T}^{T_p}(F)(g) = \bigotimes_{q\neq p} F_q(g_g) \otimes \left(\sum_{x_p \in \mathcal{Q}_p/\mathcal{K}_p} T_p(x_p)F_p(g_p x_p) \right). \tag{6.15}$$

Since $T_p \in \mathcal{H}(\mathbb{Q}_p//\mathcal{K}_p, \omega_p)$ has compact support, the sum in (6.15) is finite. It can be verified by means of Theorem 6.5, i), and Definition 5.1, ii), that (6.15) and therefore (6.14) is independent of the representative $x_p \in \mathcal{Q}_p/\mathcal{K}_p$ and thus well defined. We will show later in the paper that $\mathcal{T}^{T_p}(F)$ is indeed contained in $A_\kappa(\omega_f)$.

ii) Let p be a prime, $T_{k,l}, T_{r,s} \in \mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ as specified in Corollary 5.8 and Theorem 5.12. Then by a straightforward calculation, using (6.15) and the bilinearity of the tensor product, we obtain

$$\mathcal{T}^{xT_{k,l}+yT_{r,s}}(F)(g) = x\mathcal{T}^{T_{k,l}}(F)(g) + y\mathcal{T}^{T_{r,s}}(F)(g)$$

for all $F \in A_k(\omega_f)$, all $g \in \mathcal{G}(\mathbb{A})$ and all $x, y \in \mathbb{C}$.

There is also a compatibility relation regarding convolution:

$$\mathcal{T}^{T_{k,l}*T_{r,s}}(F)(g) = \bigotimes_{q\neq p} F_q(g_q) \otimes \left(\sum_{x_p \in \mathcal{Q}_p/\mathcal{K}_p} \left(\sum_{y_p \in \mathcal{Q}_p/\mathcal{K}_p} T_{k,l}(y_p) \circ T_{r,s}(y_p^{-1}x_p) \right) (F_p(g_p x_p)) \right).$$

Here $T_{k,l} * T_{r,s}$ is the convolution of $T_{k,l}$ and $T_{r,s}$ (cf. (5.1)). Since both sums over $\mathcal{Q}_p/\mathcal{K}_p$ are finite, we can change their order and obtain

$$\begin{aligned} &\bigotimes_{q\neq p} F_q(g_q) \otimes \left(\sum_{y_p \in \mathcal{Q}_p/\mathcal{K}_p} \left(\sum_{x_p \in \mathcal{Q}_p/\mathcal{K}_p} T_{k,l}(y_p) \circ T_{r,s}(y_p^{-1}x_p)(F_p(g_p x_p)) \right) \right) \\ &= \bigotimes_{q\neq p} F_q(g_q) \otimes \left(\sum_{z_p \in \mathcal{Q}_p/\mathcal{K}_p} T_{r,s}(z_p) \left(\sum_{y_p \in \mathcal{Q}_p/\mathcal{K}_p} T_{k,l}(y_p)(F_p(g_p y_p z_p)) \right) \right) \\ &= \mathcal{T}^{T_{r,s}} \circ \mathcal{T}^{T_{k,l}}(F)(g), \end{aligned}$$

where we have made the substitution $z_p = y_p^{-1}x_p$ in the second last equation and used the fact that $T_{k,l} \circ T_{r,s} = T_{r,s} \circ T_{k,l}$ for all $T_{k,l}, T_{r,s} \in \mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$.

Lemma 6.8. *Let p be a prime, $T_{k,l} \in \mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ as given in Corollary 5.8 and Theorem 5.12 and $F \in A_\kappa(\omega_f)$. Then $\mathcal{T}^{T_{k,l}}$*

i) *is $\mathcal{G}(\mathbb{Q})$-invariant and*
ii) *fulfils*

$$\mathcal{T}^{T_{k,l}}(F)(gk) = \omega_f^{-1}(k)\mathcal{T}^{T_{k,l}}(F)(g) \text{ for all } k \in \mathcal{K} \text{ and all } g \in \mathcal{G}(\mathbb{A}),$$
$$\mathcal{T}^{T_{k,l}}(F)(zg) = \omega_f^{-1}(z_f)\mathcal{T}^{T_{k,l}}(F)(g) \text{ for all } z \in \mathcal{Z}(\mathbb{A}) \text{ and all } g \in \mathcal{G}(\mathbb{A}).$$

Proof. i) Since $F \in A_\kappa(\omega_f)$ is $\mathcal{G}(\mathbb{Q})$-invariant, the same holds for $\mathcal{T}^{T_{k,l}}(F)$ as can be seen in (6.15).

ii) By Theorem 6.5, i), and Definition 5.1, ii) we have

$$\begin{aligned}
\mathcal{T}^{T_{k,l}}(F)(gk) &= \bigotimes_{q\neq p} F_q(g_qk_q) \otimes \left(\sum_{x_p\in\mathcal{Q}_p/\mathcal{K}_p} T_{k,l}(x_p)(F_p(g_pk_px_p)) \right) \\
&= \bigotimes_{q\neq p} \omega_q^{-1}(k_q)F_q(g_q) \otimes \left(\sum_{y_p\in\mathcal{Q}_p/\mathcal{K}_p} T_{k,l}(k_p^{-1}y_p)(F_p(g_py_p)) \right) \\
&= \begin{cases} \bigotimes_{q\neq p} \omega_q^{-1}(k_q)F_q(g_q) \otimes \omega_p^{-1}(k_p) \left(\sum_{y_p\in\mathcal{Q}_p/\mathcal{K}_p} T_{k,l}(y_p)(F_p(g_py_p))\right), & \text{if } p||D| \\ \bigotimes_{q\neq p} \omega_q^{-1}(k_q)F_q(g_q) \otimes \left(\sum_{y_p\in\mathcal{Q}_p/\mathcal{K}_p} T_{k,l}(y_p)(F_p(g_py_p))\right), & \text{if } p\nmid |D| \end{cases} \\
&= \omega_f^{-1}(k)\mathcal{T}^{T_{k,l}}(F)(g).
\end{aligned}$$

For the second equation we used the substitution $y_p = k_px_p$. This settles the first claimed identity.

For the second identity we make use of the fact that $F \in A_\kappa(\omega_f)$ and that $\omega_p(z_p)$ acts for $z_p \in \mathcal{Z}(\mathcal{K}_p)$ on S_{L_p} by multiplication with a scalar (cf. (4.14)), which commutes with the operator $T_{k,l}$. Let $z = z_\mathbb{Q}(z_\infty \times z_f)$ with $z_f = (z_q)_{q<\infty} \in \mathcal{K}$. Then

$$\begin{aligned}
\mathcal{T}^{T_{k,l}}(F)(zg) &= \bigotimes_{q\neq p} \omega_q^{-1}(z_q)F_q(g_q) \otimes \left(\sum_{x_p\in\mathcal{Q}_p/\mathcal{K}_p} T_{k,l}(x_p)(\omega_p(z_p)^{-1}F_p(g_px_p)) \right) \\
&= \omega_f^{-1}(z_f)\mathcal{T}^{T_{k,l}}(F)(g).
\end{aligned}$$

□

Let p be a prime, k, l integers with $k \leq l$ and $k + l \in 2\mathbb{Z}$, $T_{k,l} \in \mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ as in Corollary 5.8 or Theorem 5.12 and $\mathcal{T}^{T_{k,l}}$ as in Definition 6.6. We now show that the map $\mathscr{A}$ commutes with the Hecke operators $\mathcal{T}^{T_{k,l}}$ and $T(m(p^{-k}, p^{-l}))$ on both sides and thereby confirm that $\mathcal{T}^{T_{k,l}}$ indeed preserves $A_\kappa(\omega_f)$. For a prime $p \nmid |D|$ this result was in principle proved by Werner (cf. [We], Theorem 53), but not in our framework and not for a general Hecke operator $T(m(p^{-k}, p^{-l}))$.

Theorem 6.9. *Let p be a prime, $k, l \in \mathbb{Z}$ with $k \leq l$ and $k + l \in 2\mathbb{Z}$. If p divides $|D|$, let $T_{k,l} \in \mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ as in Corollary 5.8. If $(p, |D|)$, let $T_{k,l} = \mathbb{1}_{\mathcal{K}_pm(p^k,p^l)\mathcal{K}_p} \mathrm{id}_{S_{L_p}} \in$*

$\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ be as in Theorem 5.12. Further, let $\mathcal{T}^{T_{k,l}}$ be as in Definition 6.6 and $T(m(p^{-k},p^{-l}))$ the Hecke operator as defined in Section 3. Then for any $f\in S_\kappa(\rho_L)$ we have

$$\mathcal{T}^{T_{k,l}}(F_f)=F_{p^{(k+l)(\frac{\kappa}{2}-1)}T(m(p^{-k},p^{-l}))f}, \tag{6.16}$$

where F_f is the automorphic form related to f via the map $\mathscr{A}$.

Proof. We know from Lemma 6.8 that for any $g=\gamma(g_\infty\times k)\in\mathcal{G}(\mathbb{A})$ we have

$$\begin{aligned}\mathcal{T}^{T_{k,l}}(F_f)(\gamma(g_\infty\times k))&=\mathcal{T}^{T_{k,l}}(F_f)((g_\infty\times 1_f)(1\times k))\\&=\omega_f^{-1}(k)\mathcal{T}^{T_{k,l}}(F_f)(g_\infty\times 1_f).\end{aligned}$$

The same holds for $F_{p^{-(\frac{l}{2}-1)}T(m(p^{-k},p^{-l}))f}$ since it is an element of $A_\kappa(\omega_f)$. Hence, it suffices to prove (6.16) for $g=g_\infty\times 1_f$.

The proof is an adaptation of the one of Lemma 3.7 in [Ge]. We have

$$\mathcal{T}^{T_{k,l}}(F_f)(g)=\sum_{x_p\in\mathcal{Q}_p/\mathcal{K}_p}R^{T_{k,l}}(\iota_p(x_p))F_f(g\iota_p(x_p)). \tag{6.17}$$

Following an idea of Gelbart we set

$$\begin{aligned}&\gamma=(x_p,\dots,x_p,\dots)\in\mathcal{G}(\mathbb{Q}),\\&k(x_p)=(x_p^{-1},\dots,x_p^{-1},1_p,x_p^{-1},\dots)\in\mathcal{K},\\&x_p^{-1}\in\mathcal{Q}_\infty,\end{aligned}$$

where the 1_p in $k(x_p)$ is at p-th place. With these notations it is easily verified that

$$\iota_p(x_p)=\gamma(x_p^{-1}\times k(x_p)).$$

Therefore, the right-hand side of (6.17) becomes

$$\sum_{x_p\in\mathcal{Q}_p/\mathcal{K}_p}R^{T_{k,l}}(\iota_p(x_p))F_f(\gamma(x_p^{-1}g_\infty\times k(x_p))).$$

Using the fact that $F_f\in A_\kappa(\omega_f)$ and equation (6.6) subsequently, we find that the latter expression is equal to

$$\begin{aligned}&\sum_{x_p\in\mathcal{Q}_p/\mathcal{K}_p}R^{T_{k,l}}(\iota_p(x_p))\omega_f^{-1}(k(x_p))F_f(x_p^{-1}g_\infty\times 1_f)\\&=\sum_{x_p\in\mathcal{Q}_p/\mathcal{K}_p}j(x_p^{-1}g_\infty,i)^{-k}\sum_{\lambda\in D}f_\lambda(x_p^{-1}g_\infty i)R^{T_{k,l}}(\iota_p(x_p))(\omega_f^{-1}(k(x_p))\varphi_\lambda).\end{aligned}$$

Decomposing $R^{T_{k,l}}$ and ω_f into its local factors, yields

$$\begin{aligned}&\sum_{x_p\in\mathcal{K}_pm(p^k,p^l)\mathcal{K}_p/\mathcal{K}_p}j(x_p^{-1}g_\infty,i)^{-k}\times\\&\sum_{\lambda\in D}f_\lambda(x_p^{-1}g_\infty i)\bigotimes_{q\neq p}\omega_q^{-1}(x_p^{-1})\varphi_q^{(\lambda_p)}\otimes T_{k,l}(x_p)(\omega_p^{-1}(1_p)\varphi_p^{(\lambda_p)}).\end{aligned} \tag{6.18}$$

To further simplify the right-hand side of (6.18), we evaluate $\omega_q^{-1}(x_p^{-1})$ and $T_{k,l}(x_p)$ on a concrete set of representatives x_p. To this end, we first assume $k < l$. It is easily seen that Lemma 13.4 of [KL] carries over to our situation. Keeping this in mind, we can conclude that

$$\begin{aligned}&\Big\{x_{s,b} = m(p^k, p^k)\left(\begin{smallmatrix} p^s & b \\ 0 & p^{l-k-s}\end{smallmatrix}\right) \mid s = 1, \dots l-k-1, b \in (\mathbb{Z}/p^s\mathbb{Z})^\times\Big\} \\ &\cup \Big\{x_b = m(p^k, p^k)\left(\begin{smallmatrix} p^{l-k} & b \\ 0 & 1\end{smallmatrix}\right) \mid b \in \mathbb{Z}/p^{l-k}\mathbb{Z}\Big\} \cup \Big\{m(p^k, p^k)m(1, p^{l-k})\Big\}\end{aligned} \tag{6.19}$$

is a set of representatives of $\mathcal{K}_p m(p^k, p^l)\mathcal{K}_p/\mathcal{K}_p$ for any prime p. We now distinguish the cases $p \mid |D|$ and $p \nmid |D|$. The latter is easier and will be postponed to the end of the proof.

The decomposition

$$x_{s,b}^{-1} = \left(\begin{smallmatrix} r & -b \\ t & p^s\end{smallmatrix}\right) m(p^{-k}, p^{-l})n_{\text{-}}(-p^{l-k-s}t) \in \Gamma m(p^{-k}, p^{-l})\Gamma \tag{6.20}$$

with $rp^s + bt = 1$ and

$$x_b^{-1} = wm(p^{-k}, p^{-l})w^{-1}n(-b) \in \Gamma m(p^{-k}, p^{-l})\Gamma \tag{6.21}$$

can easily be verified. Since $\Gamma \subset \mathcal{K}_q$ for all primes q, these decomposition can also be interpreted as decomposition in $\mathcal{K}_q m(p^{-k}, p^{-l})\mathcal{K}_q$ for all primes q. If $q \neq p$, we may utilize Definition 4.3 and obtain

$$\begin{aligned}\omega_q^{-1}(x_{s,b}^{-1}) &= \omega_q^{-1}(n_{\text{-}}(-p^{l-k-s}t))\omega_q^{-1}(m(p^{-k}, p^{-l}))\omega_q^{-1}(\left(\begin{smallmatrix} r & -b \\ t & p^s\end{smallmatrix}\right)), \\ \omega_q^{-1}(x_b^{-1}) &= \omega_q^{-1}(w^{-1}n(-b))\omega_q^{-1}(m(p^{-k}, p^{-l}))\omega_q^{-1}(w).\end{aligned} \tag{6.22}$$

By Definition 5.1, we further have

$$\begin{aligned}T_{k,l}(x_{s,b}) &= \omega_p(n_{\text{-}}(-p^{l-k-s}t)^{-1}) \circ T_{k,l}(m(p^k, p^l)) \circ \omega_p(\left(\begin{smallmatrix} r & -b \\ t & p^s\end{smallmatrix}\right)^{-1}), \\ T_{k,l}(x_b) &= \omega_p((w^{-1}n(-b))^{-1}) \circ T_{k,l}(m(p^k, p^l)) \circ \omega_p(w^{-1}).\end{aligned} \tag{6.23}$$

Following the proof of Theorem 5.9, i), we obtain

$$\omega_q^{-1}(m(p^{-k}, p^{-l}))\varphi_q^{(\mu_q)}) = \frac{g(D_q)}{g_{p^l}(D_q)}\varphi_q^{(p^{(l-k)/2}\mu_q)}.$$

Moreover, comparing (5.7) with (4.16), it becomes apparent that the identity

$$T_{k,l}(m(p^k, p^l))\varphi_p^{(\mu_p)} = \omega_p^{-1}(m(p^{-k}, p^{-l}))\varphi_p^{(\mu_p)}$$

holds. Replacing $\omega_q^{-1}(x_p)$ and $T_{k,l}(x_p)$ in (6.18) with the expressions calculated before and piecing together the local Weil representations, we arrive at

$$\omega_f^{-1}(m(p^{-k}, p^{-l}))\varphi^{(\lambda)} = \frac{g(D)}{g_{p^l}(D)}\varphi^{(p^{(l-k)/2}\lambda)} \tag{6.24}$$

and

$$\mathcal{T}^{T_{k,l}}(F)(g_\infty \times 1_f) = \sum_{x_p \in \mathcal{K}_p m(p^k, p^l)\mathcal{K}_p/\mathcal{K}_p} j(x_p^{-1}g_\infty, i)^{-k} \sum_{\lambda \in D} f_\lambda(x_p^{-1}g_\infty i)\omega_f^{-1}(x_p^{-1})\varphi^{(\lambda)}. \tag{6.25}$$

On the other hand, it is well known that

$$\{x_{s,b}^{-1} \mid s = 1, \dots l-k-1, b \in (\mathbb{Z}/p^s\mathbb{Z})^\times\} \cup \{x_b^{-1} \mid b \in \mathbb{Z}/p^{l-k}\mathbb{Z}\} \cup \{m(p^k, p^l)^{-1}\}$$

is a set of representatives of $\Gamma/\Gamma m(p^{-k}, p^{-l})\Gamma$. In view of (6.20) and (6.21) we find

$$\rho_L^{-1}(x_{s,b}^{-1}) = \rho_L^{-1}(n_-(-p^{l-k-s}t))\rho_L^{-1}(m(p^{-k},p^{-l}))\rho_L^{-1}(\left(\begin{smallmatrix} r & -b \\ t & p^s \end{smallmatrix}\right)),$$
$$\rho_L^{-1}(x_b^{-1}) = \rho_L^{-1}(w^{-1}n(-b))\rho_L^{-1}(m(p^{-k},p^{-l}))\rho_L^{-1}(w),$$

where $\rho_L^{-1}(m(p^{-k},p^{-l}))\mathfrak{e}_\lambda = \frac{g(D)}{g_{p^l}(D)}\mathfrak{e}_{p^{(l-k)/2}\lambda}$.

Thus, taking (6.24) and (4.9) into account, we find that the right-hand side of (6.25) equals

$$j(g_\infty,i)^{-k}\sum_{x\in\Gamma/\Gamma m(p^{-k},p^{-l})\Gamma} j(x,g_\infty i)^{-k}\sum_{\lambda\in D} f_\lambda(x(g_\infty i))\rho_L^{-1}(x)\mathfrak{e}_\lambda$$
$$= F_{(p^{2l})^{1-k/2}T(m(p^{-k},p^{-l}))f}(g_\infty\times 1_f).$$

For $k=l$ the set $\mathcal{K}_p m(p^k,p^k)\mathcal{K}_p/\mathcal{K}_p$ consists only of the element $m(p^k,p^k)$. The Hecke operator $T(m(p^{-k},p^{-k}))$ acts just by multiplication with $\frac{g(D)}{g_{p^k}(D)}$. On the other hand,

$$T_{k,k}(m(p^k,p^k)) = T_k(m(p^k,p^k)) = \frac{g(D)}{g_{p^k}(D)}\,\mathrm{id}_{S_{L_p}}$$

and the arguments as before show that (6.18) equals $\frac{g(D)}{g_{p^k}(D)}F_f$.

The proof for $p\nmid |D|$ starts again with (6.18). It was already noted above that if x_p runs through the set of right cosets (6.19), then x_p^{-1} runs through a set representatives of $\Gamma/\Gamma m(p^{-k},p^{-l})\Gamma$. By Theorem 5.12 we know

$$T_{k,l}(x_p)(\omega_p^{-1}(1_p)\varphi_p^{(\lambda_p)}) = \omega_p^{-1}(1_p)\varphi_p^{(\lambda_p)}$$
$$= \varphi_p^{(\lambda_p)}.$$

Therefore,

$$\bigotimes_{q\neq p}\omega_q^{-1}(x_p^{-1})\varphi_q^{(\lambda_p)}\otimes T_{k,l}(x_p)(\omega_p^{-1}(1_p)\varphi_p^{(\lambda_p)}) = \omega_f^{-1}(x_p^{-1})\varphi^{(\lambda)}.$$

The same thoughts as before then lead again to the claimed result. The proof for $k=l$ works the same way as in the corresponding case for $p\mid |D|$. ☐

Remark 6.10. i) The identity (6.16) can be rephrased with the help of the isomorphism $\mathscr{A}$. Let $F\in A_\kappa(\omega_f)$ with the associated modular form $f_F\in S_\kappa(\rho_L)$ and $f_{\mathcal{T}^{T_{k,l}}(F)}$ the modular form corresponding to $\mathcal{T}_{T_{k,l}}(F)$. Then (6.16) is equivalent to

$$f_{\mathcal{T}^{T_{k,l}}(F)} = p^{(k+l)(\kappa/2)-1}T(m(p^{-k},p^{-l}))(f_F). \tag{6.26}$$

ii) It is also an immediate but important consequence of Theorem 6.9 and (3.11) that $f\in S_\kappa(\rho_L)$ is a common eigenform for all Hecke operators $T(m(p^k,p^l))$ if and only if the associated automorphic form F_f is a common eigenform for all operators $\mathcal{T}^{T_{k,l}}$ for all primes p and all $T_{k,l}\in\mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ ($\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ if p and $|D|$ are coprime). Remark 6.7, ii) allows us to extend this statement to the whole Hecke algebra $\mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ (and $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$). Thus,

$$\mathcal{T}^T(F_f) = \lambda_{F_f,p}(T)F_f$$

for all $T \in \mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ if and only if $f \in S_\kappa(\rho_L)$ is a common Eigenform for all Hecke operators $T(m(p^{-k}, p^{-l}))$. As in the classical scalar valued theory, we may then conclude that the map

$$\lambda_{F,p} : \mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p, \omega_p) \to \mathbb{C}, \quad T \mapsto \lambda_{F,p}(T)$$

associated to an eigenform $F \in A_\kappa(\omega_f)$ defines an algebra homomorphism.

7. Standard L-function of a common Hecke eigenform

This chapter is concerned with several issues regarding a standard L-function of a vector valued Hecke eigenform. First, we will motivate and define a standard L-function $L(s, F)$ attached to a vector valued automorphic form F. We follow the approach of Bouganis and Marzec ([BM]), Böcherer and Schulze-Pillot ([BoSP]) and Shimura ([Sh]) adapted to our situation. Via the correspondence in Theorem 6.9, it is then possible to associate the same L-function to the corresponding Hecke eigenform f_F. Along the way we establish most of the results to prove a relation between the standard zeta function $\mathcal{Z}(s, f)$ and $L(s, f)$. Afterwards, based on this relation, we prove that the introduced standard L-function can be continued meromorphically to the whole s-plane.

We assume in the whole chapter that L_p'/L_p is anisotrophic. We adopt the notation of Section 5.

7.1. Standard L-function of a vector valued automorphic form.

This section provides the necessary theory to define a standard L-function for a vector valued automorphic form as defined before. This is essentially the well known theory of spherical functions as it appeared in many places (see e. g. [Ca], [McD] or [Sa]). Here we largely follow [Ar], Chapter 5, and translate several statements therein to our setting. For primes p which are coprime to $|D|$ the proofs carry over almost verbatim. In the case of primes p dividing $|D|$ the Hecke algebra $\mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ is more complicated due to the Weil representation, causing serious difficulties. The established results then provide the means to associate an L-function $L(s, F)$ to a vector valued automorphic form F along the lines of [BM], Chapter 7.2.

Lemma 7.1. *Let p be a prime.*

i) *If p is coprime to $|D|$, any $\mathbb{C}$-algebra homomorphism $\xi : \mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p) \to \mathbb{C}$ is of the form*

$$T \mapsto \xi(T) = \widehat{\chi}_S(T), \tag{7.1}$$

Here χ is some uniquely determined unramified character of $\mathcal{M}_p$ and

$$\widehat{\chi}_S(T) = \sum_{(k,l)\in\mathbb{Z}^2} S(\langle T, \varphi_p^{(0)}\rangle)(m(p^k, p^l))\chi(m(p^k, p^l)), \tag{7.2}$$

S being the classical Satake map.

ii) *If p divides $|D|$, any $\mathbb{C}$-algebra homomorphism $\xi : \mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p, \omega_p) \to \mathbb{C}$ is of the form*

$$\begin{aligned} T \mapsto \xi(T) = \widehat{\chi}_{\mathcal{S}}(T) &= \sum_{(k,l)\in\mathbb{Z}^2} \langle (I_{\chi_{D_p}} \circ \mathcal{S}(T))(m(p^k, p^l)), \varphi_p^{(0)}\rangle\chi(m(p^k, p^l)) \\ &= \sum_{(k,l)\in\mathbb{Z}^2} \langle \mathcal{S}(T)(m(p^k, p^l)), \varphi_p^{(0)}\rangle\chi(m(p^k, p^l)), \end{aligned} \tag{7.3}$$

where $\mathcal{S}$ is the Satake map (5.14), $I_{\chi_{D_p}}$ is the isomorphism in (5.13) and χ is again an unramified character of $\mathcal{M}_p$.

Proof. First, recall that $\widehat{\chi}_S$ and $\widehat{\chi}_{\mathcal{S}}$ are well defined since $S(\langle T, \varphi_p^{(0)}\rangle)$ and $\langle \mathcal{S}(T), \varphi_p^{(0)}\rangle$ have finite support on $\mathcal{M}_p$.

i) As already noted in Theorem 5.12, $f \mapsto \langle f, \varphi_p^{(0)}\rangle$ is an isomorphism of the Hecke algebras $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ and $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p)$. From [Ca], Corollary 4.2, we know that any algebra homomorphism of $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p)$ is given by

$$g \mapsto \int_{t\in\mathcal{M}_p} S(g)(t)\chi(t)dt,$$

where χ is an unramified character of $\mathcal{M}_p$. Since $S(g)$ is bi-invariant under $\mathcal{D}_p$ and χ unramified, we obtain the above stated term. (note that we adopted the normalisation of the Haar measures in [Ca], see Section 2).

ii) The proof in i) essentially relies on the fact that any $\mathbb{C}$-algebra homomorphism ξ of the group algbra $\mathbb{C}[\mathcal{M}_p/\mathcal{D}_p]$ can be written in terms of a uniquely determined unramified character χ of $\mathcal{M}_p$ by

$$\xi(T) = \sum_{m\in\mathcal{M}_p/\mathcal{D}_p} T(m)\chi(m).$$

By Theorems 5.11 and 5.9 we know that

$$T \mapsto \langle I_{\chi_{D_p}} \circ \mathcal{S}(T), \varphi_p^{(0)}\rangle$$

maps $\mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ isomorphically to a subalgebra of $\mathbb{C}[\mathcal{M}_p/\mathcal{D}_p]$ inducing the claimed form of ξ. □

Note that each unramified character χ of $\mathcal{M}_p$ is of the form $\chi(m(t_1, t_2)) = \chi_1(t_1)\chi_2(t_2)$, where χ_i is an umramified character of $\mathbb{Q}_p^\times$, that is, χ_i is trivial on $\mathbb{Z}_p$.

Now we introduce for an unramified character χ of $\mathcal{M}_p$ the zonal spherical function on $\mathcal{Q}_p$ (cf. e. g. [Ca], p. 150): Let

$$\phi_\chi : \mathcal{Q}_p \to \mathbb{C}, \quad g = ntk \mapsto \phi_\chi(mnk) = (\chi\delta^{\frac{1}{2}})(m), \tag{7.4}$$

where $n \in N(\mathbb{Z}_p)$, $m \in \mathcal{M}_p$ and $k \in \mathcal{K}_p$. Here $\delta(m(t_1, t_2)) = \left|\frac{t_1}{t_2}\right|_p$ is the modulus character, which we already introduced in Remark 5.10. Then the zonal spherical function ω_χ on $\mathcal{Q}_p$ is defined by

$$\omega_\chi : \mathcal{Q}_p \to \mathbb{C}, \quad g \mapsto \omega_\chi(g) = \int_{\mathcal{K}_p} \phi_\chi(kg)dk, \tag{7.5}$$

where dk is the Haar measure on $\mathcal{K}_p$ normalized by $\int_{\mathcal{K}_p} dk = 1$. It follows from its definition (7.5) and (7.4) that ω_χ is bi-invariant under $\mathcal{K}_p$. The next fact is in principle well known for primes p coprime to $|D|$.

Lemma 7.2. *Let p be a prime, $T \in \mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ and*

$$\kappa_p = \frac{pg(D_p)^{-1} + 1}{p + 1}.$$

Then the following identities hold:

$$\int_{\mathcal{Q}_p} \langle T(g), \varphi_p^{(0)} \rangle \omega_\chi(g) dg = \widehat{\chi}_S(T) \tag{7.6}$$

if p *is coprime to* $|D|$.

$$\frac{1}{\kappa_p} \int_{\mathcal{Q}_p} \langle T(g)|_{S^{N(\mathbb{Z}_p)}}, \varphi_p^{(0)} \rangle \omega_\chi(g) dg = \widehat{\chi}_{\mathcal{S}}(T) \tag{7.7}$$

if p *is a divisor of* $|D|$.

Proof. If $(p, |D|) = 1$, the algebra $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ is isomorphic to the classical algebra $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p)$. In particular, T is bi-invariant with respect to $\mathcal{K}_p$. Therefore, the result can be proved as in [McD], p. 46, or [Ca], p. 150. Following the proof of either of the cited sources, we end up with

$$\int_{m \in \mathcal{M}_p} \chi(m) \delta(m)^{1/2} \int_{N(\mathbb{Q}_p)} \langle T(mn), \varphi_p^{(0)} \rangle dn. \tag{7.8}$$

Since $S(\langle T, \varphi_p^{(0)} \rangle)$ is bi-invariant under $\mathcal{D}_p$ and χ is unramified, this is equal to

$$\sum_{(k,l) \in \mathbb{Z}^2} \chi(m(p^k, p^l)) S(\langle T, \varphi_p^{(0)} \rangle)(m(p^k, p^l)).$$

For $p \mid |D|$ the computations are more involved since $T \in \mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ is not bi-invariant under $\mathcal{K}_p$. However, it is still possible to remedy the absence of the bi-invariance. To this end, we use two facts. First, due to the decomposition (7.10) and equation (7.11) below, we infer that the Weil representation $\omega_p(k)$ acts on $S_{L_p}^{\mathbb{Z}_p}$ by multiplication with the quadratic character χ_{D_p} for $k \in \mathcal{K}_0(p)$. Secondly, by Lemma 13.1 of [KL], we know that $\{T^{-j} w \mid j \in \mathbb{Z}/p\mathbb{Z}\} \cup \{1_2\}$ provides a set of coset representatives of $\mathcal{K}_p/\mathcal{K}_0(p)$. Thus, taking this and the Iwasawa decomposition into account, it follows

$$\begin{aligned} &\int_{\mathcal{Q}_p} \langle T(g)|_{S^{N(\mathbb{Z}_p)}}, \varphi_p^{(0)} \rangle \omega_\chi(g) dg \\ &= \int_{\mathcal{M}_p} \int_{N(\mathbb{Q}_p)} \int_{\mathcal{K}_0(p)} \sum_{j \in \mathbb{Z}/p\mathbb{Z}} \langle T(mn) \omega_p(T^{-j} w) \omega_p(k) \varphi_p^{(0)}, \varphi_p^{(0)} \rangle \delta(m)^{1/2} \chi(m) dm\, dn\, dk. \end{aligned} \tag{7.9}$$

For further evaluation of the integral over $\mathcal{K}_0(p)$, we remark that any matrix $k = \left(\begin{smallmatrix} a & b \\ pc & d \end{smallmatrix}\right) \in \mathcal{K}_0(p)$ can be written as product

$$k = m(\det(k), 1) m(d^{-1}) n(b) n_{-}(pcd^{-1}), \tag{7.10}$$

which allows to conclude that $\mathcal{K}_0(p)$ is equal to the direct product

$$M(\mathbb{Z}_p^\times) D((\mathbb{Z}_p^\times)^2) N(\mathbb{Z}_p) U(p\mathbb{Z}_p).$$

We may therefore replace

$$\int_{\mathcal{K}_0(p)} \text{ with } \int_{M(\mathbb{Z}_p^\times)} \int_{D(\mathbb{Z}_p^2)} \int_{N(\mathbb{Z}_p)} \int_{U(p\mathbb{Z}_p)}$$

by Fubini's theorem. For the sake of clarity we write $\int_{\mathcal{K}'}$ for the integrals $\int_{D(\mathbb{Z}_p^2)}\int_{N(\mathbb{Z}_p)}\int_{U(p\mathbb{Z}_p)}$. Since the level of L_p'/L_p is p, we have in view of (4.8) and Theorem 5.9, i)

$$\omega_p(m(d^{-1})m(ad-bpc,1)n(b)n_{\text{-}}(pcd^{-1}))\varphi_p^{(0)} = \chi_{D_p}(d)\varphi_p^{(0)} \tag{7.11}$$

and

$$T(m(p^k s_1, p^l s_2)n)\varphi_p^{(0)} = \chi_{D_p}(s_2)T(m(p^k,p^l)n)\varphi_p^{(0)}. \tag{7.12}$$

Thus, taking (7.11) and (7.12) into account, the right-hand side of (7.9) equals

$$\sum_{(k,l)\in\mathbb{Z}^2}\int_{\mathcal{D}_p}\int_{N(\mathbb{Q}_p)}\int_{M(\mathbb{Z}_p^\times)}\int_{\mathcal{K}'}\times$$
$$\sum_{j\in\mathbb{Z}/p\mathbb{Z}}\chi_{D_p}(s_2)\chi_{D_p}(r)\langle T(m(p^k,p^l)n)\omega_p(T^{-j}w)\varphi_p^{(0)},\varphi_p^{(0)}\rangle\delta(m(p^k,p^l))^{1/2}\chi(m(p^k,p^l))ds\,dn\,dr\,dk$$
$$=\sum_{(k,l)\in\mathbb{Z}^2}\int_{\mathcal{D}_p}\int_{N(\mathbb{Q}_p)}\int_{M(\mathbb{Z}_p^\times)}\int_{\mathcal{K}'}\times$$
$$\sum_{j\in\mathbb{Z}/p\mathbb{Z}}\langle T(m(p^k,p^l)n)\omega_p(T^{-j}w)\varphi_p^{(0)},\varphi_p^{(0)}\rangle\delta(m(p^k,p^l))^{1/2}\chi(m(p^k,p^l))ds\,dn\,dr\,dk.$$

The last equation results from the transformation

$$(m(s_1,s_2),n,m(r^{-1}),k)\mapsto(m(s_1,s_2)m(r^{-1}),n,m(r^{-1})m(s_2^{-1}),k).$$

In light of the assumption $\mu(\mathcal{D}_p)=1$, we may write for the last expression above

$$\mu(\mathcal{K}_0(p))\Bigg\{\sum_{(k,l)\in\mathbb{Z}^2}\delta(m(p^k,p^l))^{1/2}\chi(m(p^k,p^l))\times$$
$$\int_{N(\mathbb{Q}_p)}\langle T(m(p^k,p^l)n)\sum_{j\in\mathbb{Z}/p\mathbb{Z}}\omega_p(T^{-j}w)\varphi_p^{(0)},\varphi_p^{(0)}\rangle dn$$
$$+\sum_{(k,l)\in\mathbb{Z}^2}\delta(m(p^k,p^l))^{1/2}\chi(m(p^k,p^l)))\int_{N(\mathbb{Q}_p)}\langle T(m(p^k,p^l)n)\varphi_p^{(0)},\varphi_p^{(0)}\rangle dn\Bigg\},$$

where $\mu(\mathcal{K}_0(p))=\frac{1}{p+1}$ (see [KL], (13.3)). The formulas (4.7) of ω_p allow us to evaluate the sum over j explicitly

$$\begin{aligned}\sum_{j\in\mathbb{Z}/p\mathbb{Z}}\omega_p(T^{-j}w)\varphi_p^{(0)} &= g(D_p)^{-1}\sum_{\gamma_p\in L_p'/L_p}\sum_{j\in\mathbb{Z}/p\mathbb{Z}}e(-jq(\gamma_p))\varphi_p^{(\gamma_p)}\\ &= pg(D_p)^{-1}\varphi_p^{(0)},\end{aligned} \tag{7.13}$$

which gives

$$\begin{aligned}&\int_{\mathcal{Q}_p} \langle T(g)|_{S^{N(\mathbb{Z}_p)}}, \varphi_p^{(0)}\rangle \phi_\chi(g)dg \\ &= \kappa_p \sum_{(k,l)\in\mathbb{Z}^2} \delta(m(p^k,p^l))^{1/2}\chi(m(p^k,p^l))) \int_{N(\mathbb{Q}_p)} \langle T(m(p^k,p^l)n)|_{S^{N(\mathbb{Z}_p)}}, \varphi_p^{(0)}\rangle dn \\ &= \kappa_p \sum_{(k,l)\in\mathbb{Z}^2} \langle \mathcal{S}(T)(m(p^k,p^l), \varphi_p^{(0)}\rangle \chi(m(p^k,p^l))\end{aligned}$$

as $T|_{S^{N(\mathbb{Z}_p)}}$ is right-invariant under $N(\mathbb{Z}_p)$. □

For $s \in \mathbb{C}$ we define $\nu_s : \mathcal{Q}_p \to \mathbb{C}$ by

$$\nu_s(k_1 g k_2) = \nu_s(g) \text{ for all } k_1, k_2 \in \mathcal{K}_p \text{ and all } g \in \mathcal{Q}_p,$$

$$\nu_s(m(p^k,p^l)) = \begin{cases} p^{-(k+l)s}, & (k,l) \in \Lambda_+, \\ 0, & \text{otherwise.} \end{cases} \tag{7.14}$$

To relate $L(s,F)$ and $\mathcal{Z}(s,f_F)$, we calculate the integral

$$\int_{\mathcal{Q}_p} \nu_s(g)\omega_\chi(g)dg$$

in two different ways. The first one is an analogue of Lemma 5.2 in [Ar].

Lemma 7.3. *Let p be a prime, χ be an unramified character of $\mathcal{M}_p$,*

$$T_{k,l} \in \begin{cases} \mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p), & p \nmid |D| \\ \mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p, \omega_p), & p \mid |D|, \end{cases}$$

$$C(L_p) = \frac{g_{p^l}(D_p)}{\kappa_p g(D_p)}$$

and

$$B_S(\chi, X) = \sum_{(k,l)\in\Lambda_+} \widehat{\chi_S}(T_{k,l})X^{k+l} \text{ and } B_{\mathcal{S}}(\chi, X) = \sum_{(k,l)\in\Lambda_+} \widehat{\chi_{\mathcal{S}}}(T_{k,l})X^{k+l}$$

where κ_p is specified in Lemma 7.2 and $\widehat{\chi_S}$, $\widehat{\chi_{\mathcal{S}}}$ are defined in Lemma 7.1. Then

i)

$$\int_{\mathcal{Q}_p} \nu_s(g)\omega_\chi(g)dg = B(\chi, p^{-s}) \tag{7.15}$$

if $p \nmid |D|$.

ii)

$$\int_{\mathcal{Q}_p} \nu_s(g)\omega_\chi(g)dg = C(L_p)B(\chi, p^{-s}) \tag{7.16}$$

if $p \mid |D|$.

Proof. For p coprime to $|D|$ the proof is essentially the same as the one of Lemma 5.2 in [Ar]. One has just to replace the term ν_s with our corresponding ν_s and φ_α with $T_{k,l}$.

Again, the proof for $p \mid |D|$ is more complicated. It uses the same ideas as the ones in the proof of Lemma 7.2 and proceeds similar to the proof of [Ar], Lemma 5.2. In light of the Cartan decomposition, we have

$$\begin{aligned}\int_{\mathcal{Q}_p} \nu_s(g)\omega_\chi(g)dg &= \sum_{(k,l)\in\mathbb{Z}^2} \int_{\mathcal{K}_p}\int_{\mathcal{K}_p} \nu_s(m(p^k,p^l))\omega_\chi(m(p^k,p^l))dk_1\, dk_2 \\ &= \sum_{(k,l)\in\Lambda_+} p^{-s(k+l)}\omega_\chi(m(p^k,p^l))\end{aligned} \tag{7.17}$$

On the other hand, for $T_{k,l} \in \mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ (with $T_{k,k} = T_k$),

$$\begin{aligned}&\sum_{(k,l)\in\Lambda_+} \int_{\mathcal{K}_p}\int_{\mathcal{K}_p} \langle T_{k,l}(k_2 m(p^k,p^l)k_1)_{|_{S^N(\mathbb{Z}_p)}}, \varphi_p^{(0)}\rangle \omega_\chi(m(p^k,p^l))dk_1\, dk_2 \\ &= \sum_{(k,l)\in\Lambda_+}\sum_{u\in\mathbb{Z}/p\mathbb{Z}}\sum_{v\in\mathbb{Z}/p\mathbb{Z}} \int_{\mathcal{K}_0(p)}\int_{\mathcal{K}_0(p)} \times \\ &\langle T_{k,l}(m(p^k,p^l))\omega_p(T^{-u}wk_1')\varphi_p^{(0)}, \omega_p(T^{-v}wk_2')\varphi_p^{(0)}\rangle\omega_\chi(m(p^k,p^l))dk_1'\, dk_2' + \\ &\sum_{(k,l)\in\Lambda_+}\int_{\mathcal{K}_0(p)}\int_{\mathcal{K}_0(p)} \langle T_{k,l}(m(p^k,p^l))\omega_p(k_1')\varphi_p^{(0)}, \omega_p(k_2')\varphi_p^{(0)}\rangle\omega_\chi(m(p^k,p^l))dk_1'\, dk_2'.\end{aligned}$$

By means of the same notations and techniques as in the proof of Lemma 7.2, we may replace the last expression with

(7.18)
$$\begin{aligned}&\sum_{(k,l)\in\Lambda_+} \int_{M(\mathbb{Z}_p^\times)}\int_{\mathcal{K}'}\int_{M(\mathbb{Z}_p^\times)}\int_{\mathcal{K}'} \chi_{D_p}(r_1)\chi_{D_p}(r_2)\times \\ &\langle T_{k,l}(m(p^k,p^l))\sum_{u\in\mathbb{Z}/p\mathbb{Z}}\omega_p(T^{-u}w)\varphi_p^{(0)}, \sum_{v\in\mathbb{Z}/p\mathbb{Z}}\omega_p(T^{-v}w)\varphi_p^{(0)}\rangle\omega_\chi(m(p^k,p^l))dr_1\, dk_1\, dr_2\, dk_2 + \\ &\sum_{(k,l)\in\Lambda_+} \int_{M(\mathbb{Z}_p^\times)}\int_{\mathcal{K}'}\int_{M(\mathbb{Z}_p^\times)}\int_{\mathcal{K}'} \chi_{D_p}(r_1)\chi_{D_p}(r_2)\times \\ &\langle T_{k,l}(m(p^k,p^l))\varphi_p^{(0)}, \varphi_p^{(0)}\rangle\omega_\chi(m(p^k,p^l))dr_1\, dk_1\, dr_2\, dk_2.\end{aligned}$$

We can get rid of the characters χ_{D_p} by employing the transformation

$$(m(r_1^{-1}), k_1, m(r_2^{-1}), k_2) \mapsto (m(r_1^{-1})m(r_2^{-1}), k_1, m(r_2^{-1})m(r_1^{-1}), k_2).$$

Then by the explicit evaluation (7.13) of the Gauss sums, (7.18) simplifies to

$$\kappa_p^2 \sum_{(k,l)\in\Lambda_+} \langle T_{k,l}(m(p^k,p^l))\varphi_p^{(0)}, \varphi_p^{(0)}\rangle\omega_\chi(m(p^k,p^l)).$$

Since $T_{k,l}(m(p^k,p^l))_{|S_{L_p}^{N(\mathbb{Z}_p)}} = \frac{g(D_p)}{g_{p^l}(D_p)} \mathrm{id}_{S_{L_p}^{N(\mathbb{Z}_p)}}$, we may replace the right-hand side of (7.17) with

$$\begin{aligned}
&\frac{g_{p^l}(D_p)}{g(D_p)} \sum_{(k,l)\in\Lambda_+} p^{-s(k+l)} \langle T_{k,l}(m(p^k,p^l))\varphi_p^{(0)}, \varphi_p^{(0)}\rangle \omega_\chi(m(p^k,p^l)) \\
&= \frac{g_{p^l}(D_p)}{\kappa_p^2 g(D_p)} \sum_{(k,l)\in\Lambda_+} p^{-s(k+l)} \int_{\mathcal{K}_p m(p^k,p^l)\mathcal{K}_p} \langle T_{k,l}(g)_{|_{S^{N(\mathbb{Z}_p)}}}, \varphi_p^{(0)}\rangle \omega_\chi(g)dg \\
&= \frac{g_{p^l}(D_p)}{\kappa_p^2 g(D_p)} \sum_{(k,l)\in\Lambda_+} p^{-s(k+l)} \int_{\mathcal{Q}_p} \langle T_{k,l}(g)_{|_{S^{N(\mathbb{Z}_p)}}}, \varphi_p^{(0)}\rangle \omega_\chi(g)dg,
\end{aligned}$$

where we have used the previous calculations for the first equation. In light of Lemma 7.2 we obtain the result. □

Observe that in view of (5.24) (which can also be applied to the easier case when p is coprime to $|D|$), it is guaranteed that the Dirichlet series $B(\chi, p^{-s})$ converges in both considered instances normally for all $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large and represents in the region of convergence a holomorphic function. The equations (7.15) and (7.16) are valid for these $s \in \mathbb{C}$ and each integral on the left-hand side of these equations is consequently a holomorphic function in s on the before mentioned region.

The next Lemma is a variant of a Theorem which is due to Murase and Sugano (see [Ar], Theorem 5.3). It connects the series $B(\chi, p^{-s})$ with the rational expression

$$\frac{1 + \chi_1(p)\chi_2(p)p^{-2s}}{(1-\chi_1(p^2)p^{-2s})(1-\chi_2(p^2)p^{-2s})}$$

attached to an unramified character $\chi = (\chi_1, \chi_2)$ of $\mathcal{M}_p$.

Lemma 7.4. *Let p be a prime, $\chi = (\chi_1, \chi_2)$ be an unramified character of $\mathcal{M}_p$. Then*

$$\int_{\mathcal{Q}_p} \nu_{s+\frac{1}{2}}(g)\omega_\chi(g)dg = \frac{1 + \chi_1(p)\chi_2(p)p^{-2s}}{(1-\chi_1(p^2)p^{-2s})(1-\chi_2(p^2)p^{-2s})}. \tag{7.19}$$

Proof. A similar formula for the group $\mathrm{GL}_2(\mathbb{Q}_p)$ appears in [Mu2], p. 263. A proof for this formula can be extracted from [Sh], Lemma 3.13. Since we work with the subgroup $\mathcal{Q}_p$, we have to adjust the proof of this Lemma.

For the convenience of the reader, we repeat the relevant steps of the proof in [Sh]. We have

$$\int_{\mathcal{Q}_p} \nu_{s+\frac{1}{2}}(g)\omega_\chi(g)dg = \int_{\mathcal{Q}_p} \nu_{s+\frac{1}{2}}(g)\phi_\chi(g)dg.$$

The right-$\mathcal{K}_p$ invariance of $\nu_{s+1/2}$ and ϕ_χ allows us to write

$$\begin{aligned}
\int_{\mathcal{Q}_p} \nu_{s+\frac{1}{2}}(g)\phi_\chi(g)dg &= \int_{\mathcal{Q}_p/\mathcal{K}_p}\int_{\mathcal{K}_p} \nu_{s+\frac{1}{2}}(gk)\phi_\chi(gk)dkdg \\
&= \int_{\mathcal{Q}_p\cap M_2(\mathbb{Z}_p)/\mathcal{K}_p} \nu_{s+\frac{1}{2}}(g)\phi_\chi(g)dg.
\end{aligned}$$

It can be checked that Lemma 3.12 of [Sh] applies to our situation. Thus,

$$\begin{aligned}\int_{\mathcal{Q}_p} \nu_{s+\frac{1}{2}}(g)\phi_\chi(g)dg &= \sum_{(k,l)\in\Lambda} p^k \delta(m(p^k,p^l))^{\frac{1}{2}}\chi(m(p^k,p^l))p^{-(s+\frac{1}{2})(k+l)} \\ &= \sum_{(k,l)\in\Lambda} \chi(m(p^k,p^l))p^{-s(k+l)}.\end{aligned}$$

In order to include the condition $k+l \in 2\mathbb{N}_0$, we split each of the sums over k and l into two sums running over odd and even integers. The last expression then becomes

$$\begin{aligned}&\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \chi_1(p)^{2k}\chi_2(p)^{2l}p^{-2ks}p^{-2ls} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \chi_1(p)^{2m+1}\chi_2(p)^{2n+1}p^{-(2m+1)s}p^{-(2n+1)s} \\ &= (1+\chi_1(p)\chi_2(p)p^{-2s})[(1-\chi_1(p^2)p^{-2s})(1-\chi_2(p^2)p^{-2s})]^{-1}.\end{aligned}$$

□

Combining Lemma 7.4 with Lemma 7.3 immediately yields

Theorem 7.5. *Let* $\chi = (\chi_1, \chi_2)$ *be an unramified character of* $\mathcal{M}_p$,

$$B_S(\chi, X) = \sum_{(k,l)\in\Lambda_+} \widehat{\chi_S}(T_{k,l})X^{k+l} \text{ and } B_{\mathcal{S}}(\chi, X) = \sum_{(k,l)\in\Lambda_+} \widehat{\chi_{\mathcal{S}}}(T_{k,l})X^{k+l}.$$

Then $B_S(\chi, p^{-s})$ *and* $B_{\mathcal{S}}(\chi, p^{-s})$ *can be written as a rational expression in* $\chi_1(p)$, $\chi_2(p)$*:*

$$\begin{aligned}B_S(\chi, p^{-s}) &= \frac{1+\chi_1(p)\chi_2(p)p^{-2s+1}}{(1-\chi_1(p^2)p^{-2s+1})(1-\chi_2(p^2)p^{-2s+1})}, \\ B_{\mathcal{S}}(\chi, p^{-s}) &= \frac{1}{C(L_p)}\frac{1+\chi_1(p)\chi_2(p)p^{-2s+1}}{(1-\chi_1(p^2)p^{-2s+1})(1-\chi_2(p^2)p^{-2s+1})}.\end{aligned} \tag{7.20}$$

Now let $F \in A_\kappa(\omega_f)$ be a common eigenform of all operators $\mathcal{T}^{T_{k,l}}$, $(k,l) \in \Lambda_+$, with eigenvalues $\lambda_{F,p}(T_{k,l})$. Then according to Remark 6.10 a $\mathbb{C}$-algebra homomorphism of $\mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ (and $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p, \omega_p)$ for $(p, |D|) = 1$) is defined for each prime p via the eigenvalues $\lambda_{F,p}$. By Lemma 7.1, $\lambda_{F,p}$ determines an unramified character $\chi_{F,p} = (\chi^{(1)}_{F,p}, \chi^{(2)}_{F,p})$ of $\mathcal{M}_p$ satisfying

$$\begin{aligned}\lambda_{F,p}(T_{k,l}) &= \begin{cases}\sum_{(r,s)\in\mathbb{Z}^2} S(\langle T_{k,l}, \varphi_p^{(0)}\rangle)(m(p^r,p^s))\chi_{F,p}(m(p^r,p^s)), & (p,|D|)=1 \\ \sum_{(r,s)\in\mathbb{Z}^2} \langle \mathcal{S}(T_{k,l})(m(p^r,p^s)), \varphi_p^{(0)}\rangle\chi_{F,p}(m(p^r,p^s)), & p \mid |D|\end{cases} \\ &= \begin{cases}\widehat{\chi_{F,p}}_S(T_{k,l}), & (p,|D|)=1 \\ \widehat{\chi_{F,p}}_{\mathcal{S}}(T_{k,l}), & p \mid |D|.\end{cases}\end{aligned} \tag{7.21}$$

According to Theorem 7.5 we have

$$\begin{aligned}\sum_{(k,l)\in\Lambda_+} \lambda_{F,p}(T_{k,l})p^{-s(k+l)} &= \sum_{(k,l)\in\Lambda_+} \widehat{\chi_{F,p}}_S(T_{k,l})p^{-s(k+l)} \\ &= \frac{1+\chi^{(1)}_{F,p}(p)\chi^{(2)}_{F,p}(p)p^{-2s+1}}{(1-\chi^{(1)}_{F,p}(p^2)p^{-2s+1})(1-\chi^{(2)}_{F,p}(p^2)p^{-2s+1})}\end{aligned}$$

if $(p,|D|)=1$ and

$$\begin{aligned}\sum_{(k,l)\in\Lambda_+}\lambda_{F,p}(T_{k,l})p^{-s(k+l)} &= \sum_{(k,l)\in\Lambda_+}\widehat{\chi_{F,p_{\mathcal{S}}}}(T_{k,l})p^{-s(k+l)}\\ &= \frac{1}{C(L_p)}\frac{1+\chi^{(1)}_{F,p}(p)\chi^{(2)}_{F,p}(p)p^{-2s+1}}{(1-\chi^{(1)}_{F,p}(p^2)p^{-2s+1})(1-\chi^{(2)}_{F,p}(p^2)p^{-2s+1})}\end{aligned}$$

if $p\mid|D|$. Following [BM], Chap. 7.2, and [BoSP], Chap. I.§2, these identities give rise to the definition of a *standard L-function* associated to F.

Definition 7.6. Let $F\in A_\kappa(\omega_f)$ be a common eigenform of all operators $\mathcal{T}^{T_{k,l}}$, $(k,l)\in\Lambda_+$. We define the standard L-function of F by

$$L(s,F)=\prod_{p<\infty}L_p(s,F) \tag{7.22}$$

with

$$L_p(s,F)=\begin{cases}\frac{1+\chi^{(1)}_{F,p}(p)\chi^{(2)}_{F,p}(p)p^{-2s+1}}{(1-\chi^{(1)}_{F,p}(p^2)p^{-2s+1})(1-\chi^{(2)}_{F,p}(p^2)p^{-2s+1})}, & (p,|D|)=1,\\[2ex] \frac{1}{C(L_p)}\frac{1+\chi^{(1)}_{F,p}(p)\chi^{(2)}_{F,p}(p)p^{-2s+1}}{(1-\chi^{(1)}_{F,p}(p^2)p^{-2s+1})(1-\chi^{(2)}_{F,p}(p^2)p^{-2s+1})}, & p\mid|D|.\end{cases} \tag{7.23}$$

Let $f\in S_\kappa(\rho_L)$ and F_f be the associated automorphic form. Based on Remark 6.10 we then define the standard L-function of f by

$$L(s,f)=L(s,F). \tag{7.24}$$

7.2. **Analytic properties of $L(s,F)$.** In this section we study the analytic properties of the L-function $L(s,F)$. It turns out that it can be continued meromorphically to the whole s-plane.

7.3. **Relation to the standard zeta-function $\mathcal{Z}(s,f)$.** The subsequent exposition is essentially due to Arakawa, [Ar], Theorem 5.5, tailored to our setting.

Let $T_{k,l}$ be the operator in Corollary 5.8. Then $\sum_{(k,l)\in\Lambda_+}T_{k,l}p^{-s(k+l)}$ converges with respect to the standard norm induced by $\langle\cdot,\cdot\rangle$ on S_{L_p} for all $\mathrm{Re}(s)>1$ and is thus a well defined element in $\mathcal{H}^+(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ (and $\mathcal{H}(\mathcal{Q}_p//\mathcal{K}_p,\omega_p)$ for p coprime to $|D|$) and $\mathcal{T}^{\sum_{(k,l)\in\Lambda_+}T_{k,l}p^{-s(k+l)}}$ also makes sense. Let $F\in A_\kappa(\omega_f)$ a common eigenform all operators $\mathcal{T}^{T_{k,l}}$ and f_F the corresponding modular form in $S_\kappa(\rho_L)$. Further, let

$$f_{\mathcal{T}^{\sum_{(k,l)\in\Lambda_+}T_{k,l}p^{-s(k+l)}}(F)}\in S_\kappa(\rho_L)$$

be related to $\mathcal{T}^{\sum_{(k,l)\in\Lambda_+}T_{k,l}p^{-s(k+l)}}(F)$ by $\mathscr{A}$. We then have by Remark 6.7

$$\mathcal{T}^{\sum_{(k,l)\in\Lambda_+}T_{k,l}p^{-s(k+l)}}=\sum_{(k,l)\in\Lambda_+}p^{-s(k+l)}\mathcal{T}^{T_{k,l}}$$

and

$$\mathscr{A}^{-1}\left(\mathcal{T}^{\sum_{(k,l)\in\Lambda_+}T_{k,l}p^{-s(k+l)}}(F)\right)=\sum_{(k,l)\in\Lambda_+}p^{-s(k+l)}f_{\mathcal{T}^{T_{k,l}}(F)}.$$

Now, because of (7.21)

$$\sum_{(k,l)\in\Lambda_+} p^{-s(k+l)}\mathcal{T}^{T_{k,l}}(F) = \left(\sum_{(k,l)\in\Lambda_+} p^{-s(k+l)}\lambda_{F,p}(T_{k,l})\right) F$$
$$= \begin{cases} \left(\sum_{(k,l)\in\Lambda_+} \widehat{\chi_{F,p}}_S(T_{k,l})p^{-s(k+l)}\right) F, & (p,|D|)=1 \\ \left(\sum_{(k,l)\in\Lambda_+} \widehat{\chi_{F,p}}_{\mathcal{S}}(T_{k,l})p^{-s(k+l)}\right) F, & p \mid |D| \end{cases}$$
$$= \begin{cases} B_S(\chi_{F,p}, p^{-s})F, & (p,|D|)=1 \\ B_{\mathcal{S}}(\chi_{F,p}, p^{-s})F, & p \mid |D|. \end{cases}$$

On the other hand, using (6.26) and (3.11) we find

$$\sum_{(k,l)\in\Lambda_+} p^{-s(k+l)} f_{\mathcal{T}^{T_{k,l}}(F)} = \sum_{(k,l)\in\lambda_+} \frac{g(D)}{g_{p^{k+l}}(L)} T(m(p^k,p^l))p^{(1-\kappa/2-s)(k+l)}(f_F)$$
$$= \mathcal{Z}_p(s+\kappa/2-1, f_F)f_F$$

since f_F is an eigenform of all Hecke operators $T(m(p^k,p^l))$ by Remark 6.10. The calculations before show

$$\sum_{(k,l)\in\Lambda_+} p^{-s(k+l)} f_{\mathcal{T}^{T_{k,l}}(F)} = \mathscr{A}^{-1}\left(\sum_{(k,l)\in\Lambda_+} p^{-s(k+l)}\mathcal{T}^{T_{k,l}}(F)\right)$$
$$= \begin{cases} B_S(\chi_{F,p}, p^{-s})f_F, & (p,|D|)=1 \\ B_{\mathcal{S}}(\chi_{F,p}, p^{-s})f_F, & p \mid |D|. \end{cases}$$

It follows

$$\mathcal{Z}_p(s+\kappa/2-1, f_F) = \begin{cases} \sum_{(k,l)\in\lambda_+} \lambda_f(m(p^k,p^l))p^{(1-\kappa/2-s)(k+l)}, & (p,|D|)=1 \\ \\ \sum_{(k,l)\in\lambda_+} \frac{g(D)}{g_{p^{k+l}}(L)}\lambda_f(m(p^k,p^l))p^{(1-\kappa/2-s)(k+l)}, & p \mid |D| \end{cases}$$
$$= \begin{cases} B_S(\chi_{F,p}, p^{-s}), & (p,|D|)=1 \\ B_{\mathcal{S}}(\chi_{F,p}, p^{-s}), & p \mid |D|. \end{cases}$$

With the help of Theorem 7.5 we finally obtain

Theorem 7.7. *Let D be an anisotropic discriminant form, $f \in S_\kappa(\rho_L)$ be a common eigenform of all Hecke operators $T(m(k,l))$, $(k,l) \in \Lambda_+$, and $F_f \in A_\kappa(\omega_f)$ the corresponding automorphic form. Then*

$$\mathcal{Z}(s+\kappa/2-1, f) = L(s, F_f). \tag{7.25}$$

Theorem 7.8. *Let $\kappa \in 2\mathbb{Z}$, $\kappa \geq 3$, satisfy $2\kappa+\mathrm{sig}(L) \equiv 0 \bmod 4$, D be an anisotropic discriminant form and $f \in S_\kappa(\rho_L)$ a common eigenform of all Hecke operators $T(m(k,l)), (k,l) \in \Lambda_+$. Then the standard L-function $L(s,f)$ can be meromorphically continued to the whole s-plane.*

Proof. The proof merely boils down to use Theorem 3.3 to establish the fact that $\mathcal{Z}(s,f)$ is meromorphic on the whole s-plane and subsequently Theorem 7.7 to transfer the analytic properties of $\mathcal{Z}(s-\kappa/2-1, f)$ to $L(s, F_f)$. □

References

[An] A. N. Andrianov, *Zeta-functions of simple algebras with non-abelian characters*, Russ. Math. Surv. **23**, no 1, (1968).

[Ar] T. Arakawa, *Jacobi Eisenstein Series and a Basis Problem for Jacobi Forms*, Comment. Math. Univ. St. Paul. **43**, no 2, 181–216, (1994).

[BP] E. M. Baruch, S. Purkait, *Hecke algebras, new vectors and new forms on* $\Gamma_0(m)$, Math. Z. **287**, 705–733, (2017).

[Bo1] S. Böcherer, *Über die Funktionalgleichung automorpher L-Funktionen zur Siegelschen Modulgruppe*, J. Reine Angew. Math. **362**, 146–168, (1985).

[Bo] R. E. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **123**, no. 3, 491–562, (1998).

[BoSP] S. Böcherer, R. Schulze-Pillot, *Siegel modular forms and Theta series attached to quaternion algebras*, Nagoya Math. J. **121**, 35 – 96, (1991).

[BM] T. Bouganis, J. Marzec, *On the analytic properties of the standard L-function attached Siegel-Jacobi modular forms*, Dok. Math. **24**, 2613-2684, (2019).

[Bo2] S. Böcherer, *Ein Rationalitätssatz für formale Heckereihen zur Siegelschen Modulgruppe*, Abh. Math. Sem. Univ. Hamburg **56**, 35–47, (1986).

[Br1] J. H. Bruinier, *Borcherds Products on* $O(2,l)$ *and Chern Classes of Heegner Divisors*, Lecture notes in mathematics, **1728**, (2002).

[Br2] J. H. Bruinier, *On the converse theorem for Borcherds Products*, J. Algebra **397**, 315–342, (2014).

[BF] J. H. Bruinier, J. Funke, *On the injectivity of the Kudla-Millson lift and surjectivity of the Borcherds lift*, in: J. Lepowski, J. McKay, M. Tuite (Eds.), Moonshine - The First Quarter Century and Beyond, Cambridge University Press, 12–39, (2010).

[BEF] J. H. Bruinier, S. Ehlen, E. Freitag, *Lattices with many Borchers products*, Math. Comp. **85**, no. 300, 1953–1981 (2016).

[BS] J. H. Bruinier and O. Stein, *The Weil representation and Hecke operators for vector valued modular forms*, Math. Z. **264**, 249–270, (2010).

[BY] J. H. Bruinier, T. Yang, *Faltings Heights of CM cycles and derivatives of L-functions*, Invent. Math. **177**, 631–681, (2009).

[BK] C. J. Bushnell and P. C. Kutzko, *The admissible dual of* $GL(N)$ *via compact open subgroups*, Annals Maht. Studies, Princeton Univ. Press (1993).

[Ca] P. Cartier, *Representations of* $\mathfrak{p}$*-adic groups: A survey*, Proc. Symp. Pure Math. **33**, 1, 111–155, (1979).

[Ch] K. Chandrasekharan, *A Course on Topological Groups*, Hundustan Book Agency, (1996).

[CS] J. H. Conway, N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Third Edition, Springer-Verlag, (1999).

[De] A. Deitmar, *Automorphic Forms*, Universitext, Springer-Verlag, (2012).

[Ge] S. S. Gelbart, *Automorphic Forms on Adele Groups*, Princeton University Press, (1975).

[He] F. Herzig, *A Satake isomorphism in characteristik* p, Compos. Math. **147**, 263–283, (2011).

[HW] E. Hewitt, K. A. Ross, *Abstract Harmonic Analysis, Vol. I*, Second Edition, Springer-Verlag, (1979).

[Ho] R. Howe, *Affine-like Hecke algebras and p-adic representation theory* in Iwahori-Hecke Algebras and their Representation Theory, Lectures given at the C.I.M.E Summer School held in Martina Franca, June 28-July 6,1999, Lecture Notes in Mathematics **1804**, Springer-Verlag)2002).

[JL] G. James, M. Liebeck, *Representations and Characters of Groups*, Second Edition, Cambridge University Press, (2001).

[Ka] S. Katok, *p-adic Analysis Compared with Real*, Student Mathematical Library, Vol. 37, AMS, (2007).

[KL] A. Knightly, C. Li, *Traces of Hecke Operators*, Mathematical Surveys and Monographs, Volume 133, American Mathematical Society, (2006).

[Ku] S. Kudla, *Integrals of Borcherds Forms*, Compos. Math. **137**, 293–349, (2003).

[McD] I. G. Macdonald, *Spherical functions on a group of p-adic type*, Publ. of the Ramanujan Inst. **2**, (1971).

[Mi] T. Miyake, *Modular Forms*, Springer-Verlag, New York, (1989).

[Mu1] A. Murase, *L-functions attached to Jacobi forms of degree n, Part I. The basic identity*, J. reine angew. Math. **401**, 122-156 (1989).

[Mu2] A. Murase, *L-functions attached to Jacobi forms of degree n, Part II. Functional equation*, Math. Ann. 290, 247–276 (1991).

[Mu] F. Murnaghan, *Representations of reductive p-adic grops*, Course notes, www.math.toronto.edu/murnaghan/courses/mat119/notes.pdf, (2009).

[No] A. Nobs, *Die irreduziblen Darstellungen* $\mathrm{GL}_2(\mathbb{Z}_p)$*, insbesondere* $\mathrm{GL}_2(\mathbb{Z}_2)$, Math. Ann. **229**, 113–133, (1977).

[NW] A. Nobs, J. Wolfahrt, *Die irreduziblen Darstellungen der Gruppen* $\mathrm{SL}_2(\mathbb{Z}_p)$*, insbesondere* $\mathrm{SL}_2(\mathbb{Z}_2)$*, II. Teil*, Comment. Math. Helvetici **39**, 491–526, (1976).

[Sch] N. Scheithauer, *The Weil Representation of* $\mathrm{SL}_2(\mathbb{Z})$ *and Some Applications*, IMNR **2009**, no. 8, 1488–1545, (2009).

[Sch1] N. Scheithauer, *Automorphic products of singular weight*, Compos. Math. **153**, no. 9, 1855-1892, (2017).

[Sa] I. Satake, *Theory of spherical functions on reductive algebraic groups over p-adic fields*, I. H. E. S. Publ. Math. **18**, 5–69, (1963).

[Se] J.-P. Serre, A Course in Arithmetic, Graduate Texts in Mathematics 7, New-York, Springer-Verlag, (1973).

[Sh] G. Shimura, *Euler Products and Eisenstein Series*, Conference Board of the Mathematical Sciences (CBMS), Number **93**, AMS, (1996).

[Sh1] G. Shimura, *Arithmeticity in the Theory of Automorphic Forms*, Mathematical Surveys and Monographs **82**, AMS, (2000).

[St] O. Stein, *Analytic properties of Eisenstein series and standard L-functions*, Nagoya Math. J. **244**, 168–203, (2021).

[St1] O. Stein, *On analytic properties of the standard zeta function attached to a vector valued modular form*, Res. Number Theory **8**, No. 4, Paper No. 86., 27 p. (2022).

[St2] O. Stein, *A converse theorem for Borcherds products and the injectivity of the Kudla-Millson theta lift*, Preprint, arXiv:2306.17660, (2023).

[Ta] T. Tamagawa, *On the* ζ*-functions of a division algebra*, Ann. of Math. (2) **77**, 387–405, (1963).

[Wa] H. Wang, *Reflective Modular Forms: A Jacobi Forms Approach*, IMRN **2021**, no. 3, 2081–2107, (2021).

[Wei] A. Weil, *Sur certains groupes d' operateurs unitaires*, Acta Math. **111**, 143–211 (1964).

[We] F. Werner, *Vector valued Hecke theory*, Ph. D. thesis, Technische Universität Darmstadt, (2014).

[We1] F. Werner, Discriminant forms and Hecke operators, Master thesis, Technische Universität Darmstadt, (2011).

[Ya] S. Yamana, *L-functions and theta correspondence for classical groups*, Invent. math. **196**, 651–732, (2014).

[Ze] S. Zemel, *A p-adic approach to the Weil representation of discriminant forms arising from even lattices*, Ann. Math. Quebec **39**, 61–89, (2015).

Fakultät für Informatik und Mathematik, Ostbayerische Technische Hochschule Regensburg, Galgenbergstrasse 32, 93053 Regensburg, Germany

Email address: oliver.stein@oth-regensburg.de