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On analytic properties of the standard zeta function attached to a vector-valued modular form

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Abstract

We proof a Garrett–Böcherer decomposition of a vector-valued Siegel Eisenstein series $E^2_{l,0}$ of genus 2 transforming with the Weil representation of $\mathrm{Sp}_2(\mathbb{Z})$ on the group ring $\mathbb{C}[(L'/L)^2]$. We show that the standard zeta function associated to a vector-valued common eigenform f for the Weil representation can be meromorphically continued to the whole s-plane and that it satisfies a functional equation. The proof is based on an integral representation of this zeta function in terms of f and $E^2_{l,0}$.

Keywords: Vector-valued modular forms for the Weil representation, Standard zeta function, Garret-Böcherer decomposition

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1 Introduction

Vector-valued modular forms transforming with the Weil representation play a prominent role in the theory of Borcherds products, see e. g. [3] or [5]: The weakly holomorphic forms of this type serve as input to the celebrated Borcherds lift, which maps them to meromorphic modular forms on orthogonal groups whose zeroes and poles are supported on special divisors and which possess an infinite product expansion. This lift has many important applications in geometry, algebra and in the theory of Lie algebras. Since their prominent appearance in the works of Borcherds and Bruinier, a lot of research regarding this type of modular forms has been done. Among other things, a theory of newforms has been developed [6]. In [9] the foundations of a theory of Hecke operators was laid. The Fourier coefficients of Eisenstein series were calculated in [8] and the analytic properties of their non-holomorphic versions were studied in detail in [27]. Also, several relations between these modular forms and scalar-valued modular forms have been established (see e. g. [25]). One important part of the theory of modular forms that has not yet been addressed (to the best of my knowledge) is their relation to L-functions. We can find in the literature several ways to associate an L-function to a modular form (see e. g. [2,4,21]). Such L-functions are central objects in number theory and are studied extensively in several aspects, among them their analytic properties. The present paper can be seen

as a contribution to these investigations. Its main objective is to examine the analytic properties of a certain *L*-function of a vector-valued Hecke eigenform. The results of this paper, in particular the analytic properties of the considered *L*-function, have some important applications. For instance, they can be employed to generalize the currently existing results on the surjectivity of the Borcherds lift and the injectivity of the Kudla-Millson theta lift along the lines of [7]. As an interesting application, it should be possible to generalize classification results on reflective modular forms using the Jacobi forms approach as discussed in [28]. This is subject of a forthcoming paper by the author.

To describe the main results more closely, we introduce some notation (see Section 2 for more details). Let *L* be a non-degenerate even lattice of type (b^+, b^-) and level *N*, equipped with a bilinear form and associated quadratic form *q*. By L' we denote the dual lattice of *L*. We assume further that the rank and the signature $\mathrm{sig}(L) = b^+ - b^-$ of *L* is even. The Weil representation $\rho_{L,1}$ associated to *L* is a unitary representation of $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ on the group ring $\mathbb{C}[L'/L]$,

$$\rho_{L,1} : \Gamma_1 \to \mathrm{GL}(\mathbb{C}[L'/L]).$$

For reasons which will become apparent later we need a more general Weil representation $\rho_{L,n}$ of the symplectic group $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$ on the group ring $\mathbb{C}[(L'/L)^n]$ (cf. Definition 3.1 in Sect. 3).

Now let $l \in \mathbb{Z}$ be even. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a vector-valued modular form of weight *l* and type $\rho_{L,1}$ for the group Γ_1 if

$$f(\gamma\tau) = (c\tau + d)^l \rho_{L,1}(\gamma)f(\tau)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and *f* is holomorphic at the cusp ∞. The space of all such functions is denoted with $M_l(\rho_{L,1})$. For the subspace of cusp forms we write $S_l(\rho_{L,1})$. For the details see Sect. 4.

As already mentioned above, in [9] a Hecke operator $T(M)$ for $M \in GL_2^+(\mathbb{Q})$ was introduced by the action of a suitable Hecke algebra. Based on this definition, it is possible to recover the main results of the classical Hecke theory. In particular, it is proved that $S_l(\rho_{L,1})$ possesses a basis of common Hecke eigenforms of all Hecke operators $T\begin{pmatrix} d^2 & 0 \\ 0 & 1 \end{pmatrix}$ for all $d \in \mathbb{N}$ coprime to the level *N* of *L*. For such an eigenform we have

$$f \mid_{l,L} T\begin{pmatrix} d^2 & 0 \\ 0 & 1 \end{pmatrix} = \lambda_d(f)f$$

for $(d, N) = 1$. The authors in [9] proposed to associate to *f* an *L*-function of the form

$$L^N(s, f) = \sum_{\substack{d \in \mathbb{N} \\ (d,N)=1}} \lambda_d(f)d^{-s}.$$

L-functions of this type were considered in many places in the literature (cf. for example [2], [1] or [4]). It was suggested in [9] to establish the usual analytic properties, that is, the meromorphic continuation to the whole complex *s*-plane and a functional equation of $L^N(s, f)$ by means of a variant of the doubling method ([2], [13] or [21]). This was the starting point of the present paper. Its main content is the proof of the meromorphic continuation and a functional equation of a modified version of $L^N(s, f)$.

More precisely, we assume that there is a common Hecke eigenform *f* for *all* Hecke operators $T\begin{pmatrix} d^2 & 0 \\ 0 & 1 \end{pmatrix}$, $d \in \mathbb{N}$. This assumption is justified if a multiplicity one theorem (see e. g. [18], § 4.6) is in place. The question under what conditions such a theorem for vector-valued modular forms for the Weil representation holds was investigated in [30]. It turns

out that a multiplicity one theorem is valid if the Weil representation decomposes into irreducible subrepresentations, each subrepresentation occurring with multiplicity one. In Remark 6.1 we discuss in more detail under what circumstances $\rho_{L,1}$ allows such a decomposition. Under these assumptions we assign to f the L-function

$$Z(s,f) = \sum_{d \in \mathbb{N}} \lambda_d(f) d^{-s}, \tag{1.1}$$

which we call according to [1] a *standard zeta function*.

To study the analytic properties of (1.1), we develop a doubling method along the lines of [2] and [13], tailored to the vector-valued setting. Note that there are papers addressing this topic for vector-valued Siegel modular forms [14,15]. However, our approach deals with $\rho_{L,1}$ as representation requiring methods specifically adapted to this representation, causing several technical difficulties along the way. By restriction to the diagonal, the doubling method connects a Siegel Eisenstein series of genus $n+m$ with Siegel Eisenstein series of genus n and m and certain Poincaré series. In our setting we obtain the following pullback formula:

Theorem 1.1 *Let $d \in \mathbb{Z}$ be a positive integer, $D = \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}$ and $E_{l,0}^{2*}$ be as in Definition 4.3. Then for all $\tau, \zeta \in \mathbb{H}$*

$$E_{l,0}^{2*}\left(\begin{pmatrix} \tau & 0 \\ 0 & \zeta \end{pmatrix}, s\right)$$
$$= E_{l,0}^{1*}(\tau, s) \otimes E_{l,0}^{1*}(\zeta, s) + \frac{e(\mathrm{sig}(L)/8)}{|L'/L|^{1/2}} \sum_{d \geq 1} \frac{g_d(L)}{g(L)} d^{-l-2s} \mathscr{P}_l^+(-\tau, \zeta, D, s).$$

Here $E_{l,0}^{2*}$ is the non-holomorphic vector-valued Siegel Eisenstein series of type $\rho_{L,2}^*$, $\mathscr{P}_L^+$ means a certain vector-valued Poincaré series attached to the discriminant form $(L'/L)^2$ (analogous to the Poincaré series in [2] - see Defitinition 4.8) and $E_{l,0}^{1*}$ is the non-holomorphic vector-valued Eisenstein series of type $\rho_{L,1}^*$, where $\rho_{L,n}^*$ denotes the dual representation of $\rho_{L,n}$. Finally, $g_d(L)$ is a Gauss sum associated to the quadratic form q on L defined by

$$g_d(L) = \sum_{\mu \in L'/L} e(dq(\mu))$$

and $g(L) = g_1(L)$.

Based on this theorem, Theorems 4.10 and 6.2, we are able to calculate a Rankin–Selberg type integral depending on the Siegel Eisenstein series $E_{l,0}^2$ and an eigenform f and to thereby express it in terms of $Z(s,f)$ along the lines of [7] (see Theorem 6.4 for details). The analytic properties of $E_{l,0}^2$ with respect to s can then be transferred to $Z(s,f)$. Note that for the Siegel Eisenstein series $E_{l,0}^n$ of genus n and type $\rho_{L,n}$ these properties were established in [27]. The functional equation for $E_{l,0}^n$ is rather complicated. Corollary 4.6 specializes it to the case $n = 2$, leading to a quite clean formula.

We finish this section with an overview of the subsequent sections: After some notations and preliminaries, we introduce in Sect. 3 the Weil representation $\rho_{L,n}$ for the Siegel modular group $\mathrm{Sp}_n(\mathbb{Z})$. We prove several relations between $\rho_{L,2}$ and $\rho_{L,1}$, which are vital for the pullback formula above. The following section deals with vector-valued Siegel modular forms with respect to $\rho_{L,n}$, the Eisenstein series $E_{l,0}^n$ and certain vector-valued Poincaré series, the main ingredients of the above stated formulas. After some general remarks, we switch to the case $n = 1$ and give a brief account to Hecke operators and

explain how to extend their definition to double cosets $\Gamma_1 \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \Gamma_1$ if d is not coprime to the level N. The second part of this section defines and studies the Eisenstein series $E_{l,0}^n$. In particular, we prove a functional equation for $n = 2$ and explain how to assign to $E_{l,0}^n$ a value on the Siegel lower half space, which is a necessary technical detail to deduce the main theorems of the paper. Finally, two vector-valued Poincaré series are introduced, one of them analogous to a series introduced in [2], the other related to the first. The main results here are the reproducing formulas in Theorem 4.10. In Sect. 5 we derive the Garrett–Böcherer pullback formula in our setting. The proof follows the one of [13]. In each relevant step the corresponding calculations for $\rho_{L,2}$ have to be included, causing several technical difficulties. The last section then collects all results established before and presents the desired analytic properties of $Z(s, f)$.

2 Notation and preliminaries

We use the symbol $e(x)$, $x \in \mathbb{C}$, as an abbreviation for $e^{2\pi i x}$. As usual, by $\bar{z}$ we mean the conjugate complex number of z. In this paper, n, m are always natural numbers. For any ring R, $M_{m,n}(R)$, R^n and $\mathrm{Sym}_n(R)$ are the set of $m \times n$ matrices, the set of row vectors of size n and the set of symmetric matrices in $M_{n,n}(R)$. We write 1_n and 0_n for the unit matrix and zero matrix of size n, respectively. Moreover, by A^t and $\mathrm{tr}(A)$ we denote the transposed matrix and the trace of A. Also, for any matrix $S \in \mathrm{Sym}_n(\mathbb{R})$ we write $S > 0$ (resp. $S \geq 0$) if S is positive definite (resp. positive semi-definite). By Sp_n we denote the symplectic group of genus n. We use the symbol Γ_n for $\mathrm{Sp}_n(\mathbb{Z})$. In particular, we have $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$. For $N \in \mathbb{N}$

$$\Gamma_n(N) = \{\gamma \in \Gamma_n \mid \gamma \equiv 1_n \bmod N\}$$

is the principal congruence subgroup of Γ_n. The subgroups

$$\Gamma_{n,r} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_n \mid a = \begin{pmatrix} a_1 & 0 \\ a_3 & a_4 \end{pmatrix}, b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, c = \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix}, d = \begin{pmatrix} d_1 & d_2 \\ 0 & d_4 \end{pmatrix} \right\}, \quad (2.1)$$

where $0 \leq r \leq n$ and $*_1 \in M_{r,r}(\mathbb{Z})$, $*_2 \in M_{r,n-r}(\mathbb{Z})$, $*_3 \in M_{n-r,r}(\mathbb{Z})$, and $*_4 \in M_{n-r,n-r}(\mathbb{Z})$ are part of the definition of Siegel Eisenstein series. As special cases we have $\Gamma_{n,n} = \Gamma_n$ and

$$\Gamma_{n,0} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_n \mid c = 0_n \right\}. \quad (2.2)$$

Later on, we will use the symbol Γ_∞ for $\Gamma_{1,0}$.

Further, by $\Gamma_1 \begin{pmatrix} 0 & d^{-1} \\ d & 0 \end{pmatrix}$ we mean the subgroup

$$\Gamma_1 \begin{pmatrix} 0 & d^{-1} \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 & d^{-1} \\ d & 0 \end{pmatrix} \Gamma_1 \begin{pmatrix} 0 & d^{-1} \\ d & 0 \end{pmatrix} \cap \Gamma_1,$$

where d is a positive integer. Note that $\Gamma_1 \begin{pmatrix} 0 & d^{-1} \\ d & 0 \end{pmatrix}$ is isomorphic to

$$\Gamma_1(d) = \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}^{-1} \Gamma_1 \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \cap \Gamma_1 \quad (2.3)$$

and as a consequence, $\Gamma_1 \begin{pmatrix} 0 & d^{-1} \\ d & 0 \end{pmatrix} \backslash \Gamma_1 \cong \Gamma_1(d) \backslash \Gamma_1$. The isomorphism is given by

$$\gamma \mapsto \ell(\gamma) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \gamma^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2.4)$$

It is immediate that ℓ is an involution.

The group $\Gamma_m \times \Gamma_n$ can be embedded into Γ_{n+m} by the map

$$\iota_{m,n} : \Gamma_m \times \Gamma_n \to \Gamma_{n+m}, \quad \iota_{m,n}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}. \tag{2.5}$$

Via $\iota_{m,n}$ we can embed Γ_m and Γ_n into Γ_{n+m}:

$$\uparrow : \Gamma_m \to \Gamma_{n+m}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\uparrow} = \iota_{m,n}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \tag{2.6}$$

$$\downarrow : \Gamma_n \to \Gamma_{n+m}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\downarrow} = \iota_{n,m}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right). \tag{2.7}$$

Denote with $\mathbb{H}_n = \{\tau \in M_{n,n}(\mathbb{C}) \mid \tau^t = \tau \text{ and } \mathrm{Im}(\tau) > 0\}$ the Siegel upper half space of genus n. It is well known that the group Γ_n acts on $\mathbb{H}_n$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = (a\tau + b) \cdot (c\tau + d)^{-1}.$$

If $n = 1$, $\mathbb{H}_n = \mathbb{H}_1$ specializes to the usual upper half plane $\mathbb{H}$. Moreover, $\mathbb{H} \times \mathbb{H}$ can be embedded into $\mathbb{H}_2$ by

$$(\tau_1, \tau_2) \mapsto \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}.$$

Let $g', g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{R})$ and $\tau \in \mathbb{H}_n$. In order to define Siegel modular forms, we also need an automorphy factor j_n: We set

$$j_n(g, \tau) = \det(c\tau + d).$$

The automorphy factor satisfies the usual cocycle relation

$$j_n(gg', \tau) = j_n(g, g'\tau) j_n(g', \tau).$$

According to [12], Satz 1.4, we have the following identity

$$\mathrm{Im}(g\tau) = (c\tau + d)^{-1} \mathrm{Im}(\tau)(\overline{c\tau + d})^{-1},$$

which yields

$$\det(\mathrm{Im}(g\tau)) = |j_n(g, \tau)|^{-2} \det(\mathrm{Im}(\tau)). \tag{2.8}$$

For later purposes we introduce the function

$$w \mapsto \varphi_{l,s}(w) = w^{-l} |w|^{-2s} \tag{2.9}$$

and denote with $|A|$ the order of any finite set A. As usual, we utilize the symbol (c, d) for the greatest common divisor of the integers c and d.

3 The Weil representation

In this section we introduce a finite dimensional representation $\rho_{L,n}$ of the symplectic modular group Γ_n. It is isomorphic to a subrepresentation of the Weil representation (see e. g. [27], Sect. 3.3) as defined originally in [29]. We then specialize to the cases of $n = 2$ and $n = 1$ and study some relations between $\rho_{L,2}$ and $\rho_{L,1}$, which will be crucial for the Garret-Böcherer decomposition of the vector-valued Siegel Eisenstein series $E_{l,0}^2$.

The following notation will be used this way throughout the whole paper: Let L be a lattice of rank m equipped with a symmetric $\mathbb{Z}$-valued bilinear form $(\cdot, \cdot)$ such that the associated quadratic form

$$q(x) = \frac{1}{2}(x, x), \quad x \in L,$$

takes values in $\mathbb{Z}$. We assume that m is even, L is non-degenerate and denote its type by (b^+, b^-) and its signature $b^+ - b^-$ by $\text{sig}(L)$. Note that $\text{sig}(L)$ is also even. We stick with these assumptions on L for the rest of this paper unless we state it otherwise. Further, let

$$L' = \{x \in V = L \otimes \mathbb{Q} \mid (x, y) \in \mathbb{Z} \quad \text{for all } y \in L\}$$

be the dual lattice of L. Since $L \subset L'$, the elementary divisor theorem implies that L'/L is a finite group. The modulo 1 reduction of both, the bilinear form $(\cdot, \cdot)$ and the associated quadratic form q, defines a $\mathbb{Q}/\mathbb{Z}$-valued bilinear form $(\cdot, \cdot)$ with corresponding $\mathbb{Q}/\mathbb{Z}$-valued quadratic form q on L'/L. For any two elements $\mu, \nu \in (L'/L)^n$ we define

$$(\mu, \nu) = ((\mu_i, \nu_j))_{i,j} \in \text{Sym}_n(\mathbb{Q}),$$
$$Q[\mu] = \frac{1}{2}(\mu, \mu) \in \text{Sym}_n(\mathbb{Q}). \tag{3.1}$$

It can be easily verified that

$$\text{tr}(\mu, \nu) = \sum_{i=1}^{n}(\mu_i, \nu_i) \tag{3.2}$$

defines a $\mathbb{Q}/\mathbb{Z}$-valued bilinear form on $(L'/L)^n$ with associated quadratic form

$$\text{tr}(Q[\mu]) = \frac{1}{2}\sum_{i=1}^{n} q(\mu_i). \tag{3.3}$$

We call $((L'/L)^n, \text{tr}(\mu, \nu))$ a finite quadratic module or a discriminant form. Furthermore, we call the discriminant form $(L'/L, (\cdot, \cdot))$ anisotropic, if $q(\mu) = 0$ holds only for $\mu = 0$.

The Weil representation $\rho_{L,n}$ is a representation on the group ring $\mathbb{C}[(L'/L)^n]$. We denote its standard basis by $\{\mathfrak{e}_\lambda\}_{\lambda \in (L'/L)^n}$. The standard scalar product on $\mathbb{C}[(L'/L)^n]$ is given by

$$\left\langle \sum_{\lambda \in (L'/L)^n} a_\lambda \mathfrak{e}_\lambda, \sum_{\lambda \in (L'/L)^n} b_\lambda \mathfrak{e}_\lambda \right\rangle_n = \sum_{\lambda \in (L'/L)^n} a_\lambda \overline{b_\lambda}. \tag{3.4}$$

As Γ_n is generated by the matrices

$$S_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}, \quad T_n(b) = \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix}, \tag{3.5}$$

where $b \in \text{Sym}_n(\mathbb{Z})$, it is sufficient to define $\rho_{L,n}$ by the action on these generators. Note that we will use these symbols in the case $n = 1$ without the subscript n. Clearly, $T(b)$ is equal to T^b in this case, where T is the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_1$.

Definition 3.1 The representation $\rho_{L,n}$ of Γ_n on $\mathbb{C}[(L'/L)^n]$, defined by

$$\rho_{L,n}(T_n(b))\mathfrak{e}_\lambda = e(\text{tr}(bQ[\lambda]))\mathfrak{e}_\lambda,$$
$$\rho_{L,n}(S_n))\mathfrak{e}_\lambda = \frac{e(-\frac{n\,\text{sig}(L)}{8})}{|L'/L|^{n/2}} \sum_{\mu \in (L'/L)^n} e(-\text{tr}(\mu, \lambda))\mathfrak{e}_\mu, \tag{3.6}$$

is called *Weil representation*.

Remark 3.2   (1)   The action of

$$m_n(a) = \begin{pmatrix} a & 0 \\ 0 & (a^{-1})^t \end{pmatrix} \in \Gamma_n \tag{3.7}$$

with $a \in \mathrm{GL}_n(\mathbb{Z})$ via the Weil representation will be needed later on. It is given by

$$\rho_{L,n}(m_n(a))\mathfrak{e}_\lambda = \chi_V(\det(a))\mathfrak{e}_{\lambda a^{-1}}, \tag{3.8}$$

where $\chi_V(\det(a)) = \det(a)^{\mathrm{sig}(L)/2}$, see [31], Def. 2.2. For further details, e. g. regarding the connection of $\rho_{L,n}$ to the Weil representation in [29], see [31] or [27], p. 10.

(ii)   We denote by N the level of the lattice L. It is the smallest positive integer such that $Nq(\lambda) \in \mathbb{Z}$ for all $\lambda \in L'$. One can prove that the Weil representation $\rho_{L,n}$ is trivial on $\Gamma_n(N)$, the principal congruence subgroup of level N, cf. [23], Theorem 2.4. Therefore, $\rho_{L,n}$ factors through the finite group

$$\Gamma_n/\Gamma_n(N) \cong \mathrm{Sp}_n(\mathbb{Z}/N\mathbb{Z}).$$

(iii)   We denote by $\rho_{L,n}^*$ the dual representation of $\rho_{L,n}$. Since the Weil representation is unitary, we obtain $\rho_{L,n}^*$ from $\rho_{L,n}$ just by replacing the quadratic module $((L'/L)^n, \mathrm{tr}(\cdot, \cdot))$ with $((L'/L)^n, -\mathrm{tr}(\cdot, \cdot))$. Therefore, any result involving $\rho_{L,n}$ carries over to $\rho_{L,n}^*$. Since $\rho_{L,n}$ is unitary with respect to $\langle \cdot, \cdot \rangle_n$, its dual representation $\rho_{L,n}^*$ is equal to the complex conjugate of $\rho_{L,n}$ (interpreted as matrices), that is,

$$\rho_{L,n}^*(g) = \rho_{L,n}^{-1}(g) = \overline{\rho_{L,n}(g)}$$

for all $g \in \Gamma_n$. In particular,

$$\rho_{L,n}^*(g)\mathfrak{e}_\lambda = \sum_{\mu \in (L'/L)^n} \langle \mathfrak{e}_\lambda, \rho_{L,n}^*(g)\mathfrak{e}_\mu \rangle_n \mathfrak{e}_\mu = \sum_{\mu \in (L'/L)^n} \overline{\langle \mathfrak{e}_\lambda, \rho_{L,n}(g)\mathfrak{e}_\mu \rangle_n} \mathfrak{e}_\mu.$$

(iv)   There is an isomorphism on Γ_n:

$$\gamma \mapsto \widetilde{\gamma} = \begin{pmatrix} -1_n & 0 \\ 0 & 1_n \end{pmatrix} \gamma \begin{pmatrix} -1_n & 0 \\ 0 & 1_n \end{pmatrix}. \tag{3.9}$$

For the generators of Γ_n we have

$$\widetilde{S_n} = S_n^{-1} \text{ and } \widetilde{T_n(b)} = T_n(b)^{-1}.$$

Since the map (3.9) is an isomorphism, it follows immediately that

$$\rho_{L,n}(\widetilde{\gamma})\mathfrak{e}_\lambda = \rho_{L,n}^{-1}(\gamma) = \rho_{L,n}^*(\gamma) \tag{3.10}$$

for any $\gamma \in \Gamma_n$, where the last equation is due to part iii) of this remark.

We now concentrate on the special cases $n = 1$ and $n = 2$. It is well known that $\rho_{L,2}$ and $\rho_{L,1}$ are closely related. To describe this relation more explicitly for some elements of Γ_n, we first recall some facts from [10], p. 647 and [26], Sect. 5, which relate the group rings $\mathbb{C}[(L'/L)^2]$ and $\mathbb{C}[L'/L]$ and as a consequence $\rho_{L,2}$ and $\rho_{L,1}$: The map

$$\mathbb{C}[(L'/L)^2] \to \mathbb{C}[L'/L] \otimes \mathbb{C}[L'/L], \quad \mathfrak{e}_{(\mu,\nu)} \mapsto \mathfrak{e}_\mu \otimes \mathfrak{e}_\nu \tag{3.11}$$

defines an isomorphism, which induces an isomorphism of the representations $\rho_{L,2}$ and $\rho_{L,1} \otimes \rho_{L,1}$. It is also a known fact that we can equip $\mathbb{C}[L'/L] \otimes \mathbb{C}[L'/L]$ with an inner

product by

$$\langle \mathfrak{e}_{\lambda_1} \otimes \mathfrak{e}_{\mu_1}, \mathfrak{e}_{\lambda_2} \otimes \mathfrak{e}_{\mu_2} \rangle_2 = \langle \mathfrak{e}_{\lambda_1}, \mathfrak{e}_{\lambda_2} \rangle_1 \cdot \langle \mathfrak{e}_{\mu_1}, \mathfrak{e}_{\mu_2} \rangle_1. \tag{3.12}$$

It is then easily checked that the scalar product (3.12) coincides with (3.4) and the map (3.11) becomes an isometry.

Finally, note that $\mathbb{C}[L'/L]$ can be embedded into $\mathbb{C}[L'/L] \otimes \mathbb{C}[L'/L]$ and therefore into $\mathbb{C}[(L'/L)^2]$ via the map

$$\mathfrak{e}_\lambda \mapsto \mathfrak{e}_\lambda \otimes \mathfrak{e}_\lambda. \tag{3.13}$$

The image of $\mathbb{C}[L'/L]$ with respect to this map is isomorphic to $\mathbb{C}[L'/L]$.

Using the formulas in (3.6) and the isomorphism (3.11), we can express $\rho_{L,2}$ on the generators of Γ_2 explicitly in terms of $\rho_{L,1}$.

Lemma 3.3 *We have the following relation between $\rho_{L,2}$ and $\rho_{L,1}$ on the generators of Γ_2 and Γ_1, respectively:*

$$\begin{aligned} \rho_{L,2}(S_2)\mathfrak{e}_{(\lambda_1,\lambda_2)} &= \rho_{L,1}(S)\mathfrak{e}_{\lambda_1} \otimes \rho_{L,1}(S)\mathfrak{e}_{\lambda_2} \\ \rho_{L,2}(T_2 \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix})\mathfrak{e}_{(\lambda_1,\lambda_2)} &= \rho_L(T^{a_1})\mathfrak{e}_{\lambda_1} \otimes \rho_L(T^{a_2})\mathfrak{e}_{\lambda_2}, \end{aligned} \tag{3.14}$$

where $\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Z})$. Clearly, an analogous formula holds for S_2^{-1}.

Proof These identities can be checked by a straightforward computation involving the formulas (3.6) for $n = 2$ and $n = 1$ and the isomorphism (3.11). □

The results of the next lemma will be crucial in the proof of a Garret-Böcherer decomposition of the Siegel Eisenstein series $E_{l,0}^2$. By identifying $\mathbb{C}[(L'/L)^2]$ with $\mathbb{C}[L'/L] \otimes \mathbb{C}[L'/L]$ they provide formulas that link $\rho_{L,2}(\gamma^\uparrow)$ and $\rho_{L,2}(\gamma^\downarrow)$ with $\rho_{L,1}(\gamma)$.

Lemma 3.4 *The following formulas for the Weil representation $\rho_{L,2}$ hold:*

(i)
$$\rho_{L,2}(S^\uparrow)\mathfrak{e}_{(\lambda_1,\lambda_2)} = \rho_{L,1}(S)\mathfrak{e}_{\lambda_1} \otimes \mathfrak{e}_{\lambda_2}, \tag{3.15}$$

$$\rho_{L,2}(S^\downarrow)\mathfrak{e}_{(\lambda_1,\lambda_2)} = \mathfrak{e}_{\lambda_1} \otimes \rho_{L,1}(S)\mathfrak{e}_{\lambda_2}, \tag{3.16}$$

$$\rho_{L,2}(T^\uparrow)\mathfrak{e}_{(\lambda_1,\lambda_2)} = \rho_{L,1}(T)\mathfrak{e}_{\lambda_1} \otimes \mathfrak{e}_{\lambda_2}, \tag{3.17}$$

and

$$\rho_{L,2}(T^\downarrow)\mathfrak{e}_{(\lambda_1,\lambda_2)} = \mathfrak{e}_{\lambda_1} \otimes \rho_{L,1}(T)\mathfrak{e}_{\lambda_2}. \tag{3.18}$$

(ii) For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\rho_{L,2}(\gamma^\uparrow)\mathfrak{e}_{(\lambda_1,\lambda_2)} = \rho_L(\gamma)\mathfrak{e}_{\lambda_1} \otimes \mathfrak{e}_{\lambda_2} \tag{3.19}$$

and

$$\rho_{L,2}(\gamma^\downarrow)\mathfrak{e}_{(\lambda_1,\lambda_2)} = \mathfrak{e}_{\lambda_1} \otimes \rho_L(\gamma)\mathfrak{e}_{\lambda_2}. \tag{3.20}$$

Proof (i) For arbitrary $a \in \mathrm{Sym}_2(\mathbb{Z})$ we use the notation

$$U_2(a) = \begin{pmatrix} 1_2 & 0 \\ a & 1_2 \end{pmatrix} \in \Gamma_2. \tag{3.21}$$

One can easily verify the decomposition

$$S^{\uparrow} = U_2(b)T_2(-b)U_2(b) \text{ and}$$
$$S^{\downarrow} = U_2(c)T_2(-c)U_2(c),$$

where $b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $c = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. The matrix $U_2(a)$ can be written as a product of the generators (3.5)

$$U_2(a) = S_2 T_2(-a) S_2^{-1}$$

and therefore

$$S^{\uparrow} = S_2 T_2(-b) S_2^{-1} T_2(-b) S_2 T_2(-b) S_2^{-1}, \quad S^{\downarrow} = S_2 T_2(-c) S_2^{-1} T_2(-c) S_2 T_2(-c) S_2^{-1}.$$

If we employ the identities (3.14) successively for all constituents of the decompositions above, (3.6) applied to $T_2(-b)$ and $T_2(-c)$, and the bilinearity of the tensor product, we may transfer the evaluation of $\rho_{L,2}(S^{\uparrow})$ to the evaluation of $\rho_{L,1}$ on the corresponding elements in Γ_1:

$$
\begin{aligned}
\rho_{L,2}(S^{\uparrow})\mathfrak{e}_{(\lambda_1,\lambda_2)} &= \rho_{L,1}(ST^{-1}S^{-1}T^{-1}ST^{-1}S^{-1})\mathfrak{e}_{\lambda_1} \otimes \rho_{L,1}(SS^{-1}SS^{-1})\mathfrak{e}_{\lambda_2} \\
&= \rho_{L,1}(ST^{-1}S^{-1}T^{-1}ST^{-1}S^{-1})\mathfrak{e}_{\lambda_1} \otimes \mathfrak{e}_{\lambda_2}
\end{aligned}
\tag{3.22}
$$

and accordingly

$$
\begin{aligned}
\rho_{L,2}(S^{\downarrow})\mathfrak{e}_{(\lambda_1,\lambda_2)} &= \rho_{L,1}(SS^{-1}SS^{-1})\mathfrak{e}_{\lambda_1} \otimes \rho_{L,1}(ST^{-1}S^{-1}T^{-1}ST^{-1}S^{-1})\mathfrak{e}_{\lambda_2} \\
&= \mathfrak{e}_{\lambda_1} \otimes \rho_{L,1}(ST^{-1}S^{-1}T^{-1}ST^{-1}S^{-1})\mathfrak{e}_{\lambda_2}.
\end{aligned}
\tag{3.23}
$$

We can further simplify the expressions above by means of [17], Lemma 4.6. It states for integers a, d satisfying $ad \equiv 1 \bmod N$ that

$$\rho_{L,1}(ST^d S^{-1} T^a ST^d)\mathfrak{e}_{\lambda} = \frac{g_d(L)}{g(L)}\mathfrak{e}_{d\lambda},$$

where

$$g_d(L) = \sum_{\lambda \in L'/L} e(dq(\lambda)) \tag{3.24}$$

is a Gauss sum (explaining the notation g_d) attached to the quadratic form q and $g(L) = g_1(L)$. If we apply this result for $a = d = -1$, we obtain

$$\rho_{L,1}(ST^{-1}S^{-1}T^{-1}ST^{-1}S^{-1})\mathfrak{e}_{\lambda} = \frac{e\left(\frac{\text{sig}(L)}{8}\right)}{\sqrt{|L'/L|}} \sum_{\mu \in L'/L} e((\mu, \lambda)) \frac{g_{-1}(L)}{g(L)}\mathfrak{e}_{-\mu}.$$

By Milgram's formula,

$$g(L) = \sqrt{|L'/L|}e(\text{sig}(L)/8),$$

we have

$$e(\text{sig}(L)/8)\frac{g_{-1}(L)}{g(L)} = e(-\text{sig}(L)/8),$$

which implies

$$\rho_{L,1}(ST^{-1}S^{-1}T^{-1}ST^{-1}S^{-1})\mathfrak{e}_\lambda = \rho_{L,1}(S)\mathfrak{e}_\lambda.$$

The identities (3.17) and (3.18) are due to Lemma 3.3 and the fact that

$$T^\uparrow = T_2(b) \text{ and } T^\downarrow = T_2(c),$$

where b, c have same meaning as before in the proof.

(ii) This follows easily from the fact that the embeddings (2.6) and (2.7) are group homomorphisms combined with the corresponding formulas for the generators S and T of Γ_1.

$\square$

We end this section with the evaluation of the Weil representation $\rho_{L,2}$ on the element $U_2(D) \in \Gamma_2$, where D is equal to $\left(\begin{smallmatrix} 0 & d \\ d & 0 \end{smallmatrix}\right)$ and d is a positive integer. This evaluation will also contribute to the proof of a Garret-Böcherer decomposition of the Siegel Eisenstein series $E^2_{l,0}$.

Lemma 3.5 *Let* $d \in \mathbb{N}$, $D = \left(\begin{smallmatrix} 0 & d \\ d & 0 \end{smallmatrix}\right) \in \mathrm{Sym}_2(\mathbb{Z})$ *and* $U_2(D) \in \Gamma_2$ *defined as in* (3.21).
Then

$$\rho_{L,2}^{-1}(U_2(D))\mathfrak{e}_{\lambda_1} \otimes \mathfrak{e}_{\lambda_2} = \frac{1}{|L'/L|} \sum_{\mu,\nu \in L'/L} e((\mu, \lambda_2 - \nu))\mathfrak{e}_{d\mu+\lambda_1} \otimes \mathfrak{e}_\nu. \tag{3.25}$$

Proof Clearly, $U_2(D)^{-1} = U_2(-D) = S_2 T_2(D) S_2^{-1}$. A straightforward calculation using Lemma 3.3 and (3.6) yields

$$\begin{aligned} &\rho_{L,2}(U_2(-D))\mathfrak{e}_{\lambda_1} \otimes \mathfrak{e}_{\lambda_2} \\ &= \frac{1}{|L'/L|^2} \sum_{\nu_1,\nu_2 \in L'/L} \sum_{\mu_2 \in L'/L} e((\mu_2, \lambda_2 - \nu_2)) \sum_{\mu_1 \in L'/L} e((\mu_1, d\mu_2 + \lambda_1 - \nu_1))\mathfrak{e}_{\nu_1} \otimes \mathfrak{e}_{\nu_2}. \end{aligned}$$

The sum over μ_1 is equal to $|L'/L|$ if $\nu_1 = d\mu_2 + \lambda_1$ and 0 otherwise. Hence,

$$\rho_{L,2}(U_2(D)^{-1})\mathfrak{e}_{\lambda_1} \otimes \mathfrak{e}_{\lambda_2} = \frac{1}{|L'/L|} \sum_{\nu_2 \in L'/L} \sum_{\mu_2 \in L'/L} e((\mu_2, \lambda_2 - \nu_2))\mathfrak{e}_{d\mu_2+\lambda_1} \otimes \mathfrak{e}_{\nu_2},$$

as claimed.

$\square$

4 Vector-valued Siegel modular forms and Eisenstein series

One major goal of the present paper is to study the analytic properties of the standard zeta function of a certain vector-valued modular form of type $\rho_{L,1}$ (a common eigenform of all Hecke operators $T(m^2)$). The main tool in this regard is the Garret-Böcherer decomposition of a non-holomorphic vector-valued Siegel Eisenstein series $E^2_{l,0}$. In this section we provide the necessary background to prove the Garret-Böcherer decomposition and to define and study the aforementioned standard zeta function. To this end, we collect some facts about vector-valued Siegel modular forms of type $\rho_{L,n}$ and introduce the non-holomorphic vector-valued Siegel Eisenstein series $E^n_{l,0}$ transforming with $\rho_{L,n}$,

which plays a key role in determining the analytic properties of the standard zeta function. As a matter of fact, the zeta function inherits these properties from $E_{l,0}^2$, which is the reason we state and discuss them in some detail in this section. We follow [31] and [27].

4.1　Vector-valued Siegel modular forms of type $\rho_{L,n}$

Let $l \in \mathbb{Z}$. For $f : \mathbb{H}_n \longrightarrow \mathbb{C}[(L'/L)^n]$ and $g \in \Gamma_n$ we define the usual *Petersson slash operator* by

$$(f \mid_{l,L}^n g)(\tau) = j_n(g, \tau)^{-l} \rho_{L,n}(g)^{-1} f(g\tau).$$

The Petersson slash operator applied to the dual Weil representation $\rho_{L,n}^*$ will be denoted with $\mid_{l,L}^{n*}$. For $n = 1$ we drop the superscript and subscript n and simply write $j(g, \tau)$, $\mid_{l,L}$ or $\mid_{l,L}^*$.

Definition 4.1　Let $l \in \mathbb{Z}$. A function $f : \mathbb{H}_n \longrightarrow \mathbb{C}[(L'/L)^n]$ is called a Siegel modular form of weight l with respect to Γ_n and $\rho_{L,n}$ if

　i)　f is holomorphic on $\mathbb{H}_n$,
　ii)　$f \mid_{l,L}^n g = f$　for all $g \in \Gamma_n$,
　iii)　f has a Fourier expansion of the form

$$f(\tau) = \sum_{\lambda \in (L'/L)^n} \sum_{\substack{t \in \Lambda_n \\ t+Q[\lambda] \geq 0}} a(\lambda, t + Q[\lambda])e(\mathrm{tr}((t + Q[\lambda])\tau))\mathfrak{e}_\lambda, \qquad (4.1)$$

where Λ_n is the set of half-integral $(n \times n)$-matrices.

We denote the space of these modular forms with $M_l(\rho_{L,n})$. Note that the vanishing condition on the Fourier coefficients iii) is automatically fulfilled for $n \geq 2$ by the Koecher principle. Moreover, if $a(\lambda, t + Q[\lambda]) = 0$ for all $t \in \Lambda_n$ with $t + Q[\lambda] = 0$, then f is called a cusp form. We use the notation $S_l(\rho_{L,n})$ for the subspace of all cusp forms.

Remark 4.2　(i)　We could have defined the modular forms in Definition 4.1 also for half integral weight l. The following argument shows that $M_l(\rho_{L,n}) = \{0\}$ if the condition

$$2l \equiv \mathrm{sig}(L) \bmod 2 \qquad (4.2)$$

is not satisfied. For $e_n = \mathrm{diag}(-1, 1, \ldots, 1) \in \mathrm{GL}_n(\mathbb{Z})$ and $f \in M_l(\rho_{L,n})$ by (3.8) we have

$$f(\tau) = (f \mid_{l,L}^n m_n(e_n))(\tau) = (-1)^{-l}(-1)^{\mathrm{sig}(L)/2} f(\tau)$$

for all $\tau \in \mathbb{H}_n$, where $m_n(e_n)$ is given by (3.7). Thus $2l$ and $\mathrm{sig}(L)$ must either be both even or odd if f is a non-trivial form in $M_l(\rho_{L,n})$. Since we have assumed that the rank of L is even, so is the signature $\mathrm{sig}(L)$ and therefore there are only non-trivial integral weight modular forms.

(ii)　Since $\rho_{L,n}$ is trivial on $\Gamma_n(N)$, it follows directly from Definition 4.1 that the component functions f_λ of $f = \sum_{\lambda \in (L'/L)^n} f_\lambda \mathfrak{e}_\lambda$ are Siegel modular forms of weight l for $\Gamma_n(N)$.

4.1.1 Petersson scalar product and Hecke operators

For $f, g \in M_l(\rho_{L,n})$, one of which is a cusp form, we define the Petersson scalar product of f and g analogously to [5], Sect. 1.2.2., by

$$(f, g)_n = \int_{\Gamma_n \backslash \mathbb{H}_n} \langle f(\tau), g(\tau) \rangle_n \det(\mathrm{Im}(\tau))^l d\mu_n(\tau). \tag{4.3}$$

Here $d\mu_n(\tau) = \det(y)^{-n-1} dx dy$ with $\tau = x + iy$ is the usual symplectic volume element and $\langle \cdot, \cdot \rangle_n$ is given in (3.4). If $n = 1$, we omit the subscript n. The same arguments as in [5] show that the integral (4.3) is well-defined.

Let $D = \left(\begin{smallmatrix} d & 0 \\ 0 & d^{-1} \end{smallmatrix} \right)$, $D' = \left(\begin{smallmatrix} d^2 & 0 \\ 0 & 1 \end{smallmatrix} \right)$ and $R_d = \left(\begin{smallmatrix} d & 0 \\ 0 & d \end{smallmatrix} \right)$. We now briefly describe the definition of Hecke operators $T(D')$ and $T(D)$ on $M_l(\rho_{L,1})$ (to the best of my knowledge there is no theory of Hecke operators for $M_l(\rho_{L,n})$ known, although it might be developed in a similar manner). We follow [9]. All details can be found therein. According to our assumptions on L, we only need to consider the case of even signature. Recall that N is the level of L. We have to distinguish between the cases $(d, N) = 1$ and $(d, N) > 1$.

Let

$$\mathcal{G}(N) = \left\{ M \in \mathrm{GL}_2^+(\mathbb{Q}) \mid \quad \exists n \in \mathbb{Z} \text{ with } (n, N) = 1 \text{ such that } nM \in M_2(\mathbb{Z}) \right. \\ \left. \text{and } (\det(nM), N) = 1 \right\}. \tag{4.4}$$

If d is coprime to N, $(D, 1)$ and (D', d) belong to the group

$$\mathcal{Q}(N) = \left\{ (M, r) \in \mathcal{G}(N) \times (\mathbb{Z}/N\mathbb{Z})^* \mid \quad \det(M) \equiv r^2 \pmod{N} \right\}. \tag{4.5}$$

We identify Γ_1 with a subgroup of $\mathcal{Q}(N)$ via the map $\gamma \mapsto (\gamma, 1)$. In [9] an extension of $\rho_{L,1}$ to the group $\mathcal{Q}(N)$ is introduced. The condition that the determinant of any element of this group is coprime to the level is crucial. Based on this extension, we now define Hecke operators in terms of the Hecke algebra given by the pair of groups $(\mathcal{Q}(N), \Gamma_1)$. For any $(g, r) \in \mathcal{Q}(N)$ and $f \in M_l(\rho_{L,1})$ we set

$$f \mid_{l,L} T(g, r) = \det(g)^{l/2-1} \sum_{M \in \Gamma_1 \backslash \Gamma_1 g \Gamma_1} f \mid_{l,L} (M, r). \tag{4.6}$$

Instead of $T(D, 1)$ and $T(D', d)$ we write $T(D)$ and $T(D')$, respectively. Taking [9], (3.5), into account, it can be easily verified that $\rho_{L,1}(D)$ and $\rho_{L,1}(D')$ are connected via

$$\rho_{L,1}(D') = \rho_{L,1}(R_d)\rho_{L,1}(D) = \frac{g(L)}{g_d(L)} \rho_{L,1}(D). \tag{4.7}$$

This immediately yields a relation between $T(D)$ and $T(D')$:

$$T(D) = \frac{g(L)}{g_d(L)} T(D'). \tag{4.8}$$

Replacing $\rho_{L,1}$ with its dual representation gives the corresponding relation

$$T(D) = \frac{g_d(L)}{g(L)} T(D'). \tag{4.9}$$

If $(d, N) > 1$, D and D' do not belong to $\mathcal{Q}(N)$ and the Weil representation cannot be extended to a suitable subgroup of $\mathrm{GL}_2^+(\mathbb{Q})$ as before. However, it can be extended to the double coset $\Gamma_1 D' \Gamma_1$ in the following way ([9], Sect. 5):

$$\rho_{L,1}^{-1}(D')\mathfrak{e}_\lambda = \mathfrak{e}_{d\lambda} \tag{4.10}$$

and for $\delta = \gamma D' \gamma' \in \Gamma_1 D' \Gamma_1$ we put

$$\rho_{L,1}^{-1}(\delta)\mathfrak{e}_\lambda = \rho_{L,1}^{-1}(\gamma')\rho_{L,1}^{-1}(D')\rho_{L,1}^{-1}(\gamma)\mathfrak{e}_\lambda. \tag{4.11}$$

Note that (4.10) is compatible with the definition of $\rho_{L,1}$ in the case of $(d, N) = 1$ (see [9], Lemma 3.6). It can be shown (see [9], Sect. 5) that (4.11) is independent of the decomposition of δ. With these definitions in mind, we define the Hecke operator $T(D')$ exactly as in (4.6). In order to define the operator $T(D)$, we set in consistency with [9], (3.5)

$$\rho_{L,1}^{-1}(R_d)\mathfrak{e}_\lambda = \frac{g_d(L)}{g(L)}\mathfrak{e}_\lambda \text{ and } \rho_{L,1}(R_d)\mathfrak{e}_\lambda = \frac{g(L)}{g_d(L)}\mathfrak{e}_\lambda. \tag{4.12}$$

Accordingly, we define

$$\rho_{L,1}^{-1}(D)\mathfrak{e}_\lambda = \frac{g_d(L)}{g(L)}\rho_{L,1}^{-1}(D')\mathfrak{e}_\lambda = \frac{g_d(L)}{g(L)}\mathfrak{e}_{d\lambda}. \tag{4.13}$$

Since the $\rho_{L,1}^{-1}(D)$ and $\rho_{L,1}^{-1}(D')$ differ just by a constant factor,

$$\rho_{L,1}^{-1}(\gamma D\gamma') = \frac{g_d(L)}{g(L)}\rho_{L,1}^{-1}(\gamma D'\gamma') \tag{4.14}$$

together with (4.11) defines an action of the double coset $\Gamma_1 D\Gamma_1$. Again, we find that the relation (4.8) holds.

Finally, note that the analogous identity to (5.5) in [9] also holds for D. It can easily be confirmed that

$$\langle \mathfrak{b}, \rho_{L,1}^{-1}(D)\mathfrak{a}\rangle_1 = \langle \rho_{L,1}^{-1}(D^{-1})\mathfrak{b}, \mathfrak{a}\rangle_1 \tag{4.15}$$

for any $\mathfrak{a}, \mathfrak{b} \in \mathbb{C}[L'/L]$.

If d is coprime to N, one can show that $\frac{g(L)}{g_d(L)}$ is real and thus $T(D)$ is self-adjoint with respect to $(\cdot, \cdot)_1$. If $(d, N) > 1$, it is in general not true that $\frac{g(L)}{g_d(L)}$ is a real number. In fact, it can be shown that it is an 8-th root of unity (see [24], Sect. 3 , for these assertions). In this case, in light of [9], Theorem 5.6, we may only conclude that

$$(f, g \mid_{l,L} T(D))_1 = \kappa_d(f \mid_{l,L} T(D), g)_1 \tag{4.16}$$

holds, where

$$\kappa_d = \overline{\frac{g(L)}{g_d(L)}}\frac{g_d(L)}{g(L)}.$$

In both cases $T(D)$ preserves the space $S_l(\rho_{L,1})$.

4.2 Vector-valued Eisenstein series

Eisenstein series and Poincaré series are important examples of the vector-valued Siegel modular forms in Definition 4.1. In this subsection we introduce non-holomorphic versions of these series and study the analytic properties of the real-analytic Eisenstein series $E_{l,0}^n$ for Γ_n transforming with $\rho_{L,n}$.

Definition 4.3 Let $l \in 2\mathbb{Z}$, $l \geq n + 1$, satisfy the condition $2l + \text{sig}(L) \equiv 0 \mod 4$. For $\tau \in \mathbb{H}_n$ and $s \in \mathbb{C}$ we define by

$$\begin{aligned}
E_{l,0}^n(\tau, s) &= \sum_{\gamma \in \Gamma_{n,0}\backslash\Gamma_n} \det(\text{Im}(\tau))^s \mathfrak{e}_0 \mid_{l,L}^n \gamma \\
&= \det(\text{Im}(\tau))^s \sum_{\gamma \in \Gamma_{n,0}\backslash\Gamma_n} |j_n(\gamma, \tau)|^{-2s} j_n(\gamma, \tau)^{-l}\rho_{L,n}^{-1}(\gamma)\mathfrak{e}_0.
\end{aligned} \tag{4.17}$$

a vector-valued non-holomorphic Siegel Eisenstein series of weight l transforming with $\rho_{L,n}$.

Remark 4.4 (i) From the formulas (3.6), (3.8) and the assumption $2l \equiv -\operatorname{sig}(L) \bmod 4$ follows that $\det(\operatorname{Im}(\tau))^s \mathfrak{e}_0$ is $\Gamma_{n,0}$-invariant with respect to the slash operator.

(ii) From [27], Lemma 3.14, it follows that $E_{l,0}^n(\tau, s)$ converges absolutely for all $\tau \in \mathbb{H}_n$ and all $s \in \mathbb{C}$ with $\operatorname{Re}(s) > \frac{n+1-l}{2}$. The usual argument then shows that $E_{l,0}^n$ is, with respect to $|_{l,L}$, a Γ_n-invariant real-analytic function. In particular for $l > n+1$, $\tau \mapsto E_{l,0}^n(\tau, 0)$ is a vector-valued Siegel modular form in $M_l(\rho_{L,n})$.

(iii) Although $E_{l,0}^n$ is defined on $\mathbb{H}_n$, we can also assign to $E_{l,0}^n(-\tau)$, $\tau \in \mathbb{H}_n$, a meaningful value: To this end, let $E_{l,0}^{n*}$ be the Eisenstein series in Definition 4.3 with respect to the dual Weil representation $\rho_{L,n}^*$. Then, since $\gamma \mapsto \widetilde{\gamma}$ is an isomorphism preserving $\Gamma_{n,0}$, we have that

$$E_{l,0}^{n*}(\tau, s) = \det(\operatorname{Im}(\tau))^s \sum_{\gamma \in \Gamma_{n,0} \backslash \Gamma_n} |j_n(\widetilde{\gamma}, \tau)|^{-2s} j_n(\widetilde{\gamma}, \tau)^{-l} \rho_{L,n}^{*-1}(\widetilde{\gamma}) \mathfrak{e}_0,$$

where $\widetilde{\gamma}$ is given by (3.9). Clearly, $j_n(\widetilde{\gamma}, \tau) = j_n(\gamma, -\tau)$ and by Remark 3.2, iv),

$$\rho_{L,n}^{*-1}(\widetilde{\gamma}) = \rho_{L,n}^{-1}(\gamma).$$

Therefore, we put

$$\begin{aligned} E_{l,0}^n(-\tau, s) &= ((-1)^n)^s \, E_{l,0}^{n*}(\tau, s) \text{ and} \\ E_{l,0}^n(\overline{\tau}, s) &= ((-1)^n)^s \, E_{l,0}^{n*}(-\overline{\tau}, s). \end{aligned} \tag{4.18}$$

With the same arguments

$$\begin{aligned} E_{l,0}^{n*}(-\tau, s) &= ((-1)^n)^s E_{l,0}^n(\tau, s) \text{ and} \\ E_{l,0}^{n*}(\overline{\tau}, s) &= ((-1)^n)^s E_{l,0}^n(-\overline{\tau}, s) \end{aligned} \tag{4.19}$$

is justified.

The analytic properties of $E_{l,0}^n$ with respect to s were investigated in [27]. The functional equation of $E_{l,0}^n$ for general n is quite complicated, yet for $n = 2$ it is much simpler and quite clean. To avoid the summary of the extensive notation in [27], we refer for the most part to [27] and content ourselves with the recapitulation of as much material as needed to understand the statement of the analytic properties of $E_{l,0}^n$. (Note that there is a misprint in [27] regarding the set

$$\{v_{i_1,\ldots,i_d}\}_k, \quad 0 \leq i_1 < i_2 < \cdots < i_d \leq k-1$$

Incorrectly, n is given there as the upper bound for the indices i_j. Whenever the indices i_j appear in the original paper, n should be replaced by $k-1$ as the upper bound for i_j.)

Afterwards, we specialize to the case of $n = 2$ and present the corresponding result and proof for $E_{l,0}^2$.

We start with the notation which is needed for the transformation behaviour regarding the primes p dividing $|L'/L|$: By S_n we mean the symmetric group acting on $I = \{1, \ldots, n\}$ and σ is an element of S_n. Moreover, $I = I_0 \cup \cdots \cup I_s$ denotes a partition of I into disjoint σ-stable subsets. Depending on σ and $0 \leq r \leq s$ we have the basic quantities

$$\begin{aligned} c_{1,r}(\sigma) &= |\{i \in I_r \mid \sigma(i) = i\}|, \\ c_{2,r}(\sigma) &= |\{i \in I_r \mid \sigma(i) > i\}|. \end{aligned} \tag{4.20}$$

They are part of several further quantities. We list here two of them: for a positive integer $k \leq n$ we put

$$A_i(\sigma) = \sum_{j=0}^{i-1}(\frac{1}{2}c_{1,j}(\sigma) + c_{2,j}(\sigma)), \quad B_i(\sigma) = -mA_i(\sigma) + n(0) - n(i), \ i = 1, \ldots, k,$$
(4.21)

where $n(0)$ and $n(i)$ are integers depending on σ and which are specified in [27], (3.38). The terms in (4.21) appear as argument of the local zeta function

$$\zeta_p(s) = (1 - p^{-s})^{-1} :$$

$$D_{p,j}(s) = \zeta_p\left(4A_j(\sigma)(s - s_0) - 2B_j(\sigma)\right)$$
(4.22)

with $s_0 = \frac{m}{2} - \frac{n+1}{2}$. Apart from these local zeta functions the two quantities

$$\kappa_p(\sigma, I, k) \text{ and } \Delta_p(\sigma, q, k)$$

contribute to the functional equation of $E_{l,0}^n$. For their definition we refer to [27], p. 184. Note that the parameter k in the symbol $\Delta_p(\sigma, q, k)$ is missing in [27], which could give the impression that it does not depend on this parameter. This is in fact not the case and the reason why we add it here.

Now we consider the contribution of the primes to the functional equation coprime to $|L'/L|$. These primes are collected in the set $P = \{q \text{ prime } \mid q \nmid |L'/L|\}$. For each such prime we put

$$a_{n,p}(s) = L_p(s - \frac{n}{2} + \frac{1}{2}, \chi_{V,p}) \prod_{k=1}^{[\frac{n}{2}]} \zeta_p(2s - n + 2k),$$

$$b_{n,p}(s) = L_p(s + \frac{n+1}{2}, \chi_{V,p}) \prod_{k=1}^{[\frac{n}{2}]} \zeta_p(2s + n - 2k + 1),$$
(4.23)

and form the products

$$a_n^P(s) = \prod_{p \in P} a_{n,p}(s), \quad b_n^P(s) = \prod_{p \in P} b_{n,p}(s).$$
(4.24)

Here $L_p(s, \chi_{V,p}))$ means the local L-function given by $(1 - \chi_{V,p}(p)p^{-s})$ and $\chi_{V,p}$ is a character given by the local Hilbert symbol (see e. g. [27], Lemma 3.4. for the details). The constant $\xi(s, n, k, P)$ then gathers the contribution of the places belonging to the primes $p \in P$ and the archimedian place:

$$\xi(s, n, k, P) = \frac{(-1)^{k/2}2^{(1-s)n}\pi^{\frac{n(n+1)}{2}}}{|L'/L|^{n/2}} \frac{\Gamma_n(s)}{\Gamma_n(\alpha)\Gamma_n(\beta)} \frac{a_n^P(s)}{b_n^P(s)},$$
(4.25)

where $\alpha = \frac{1}{2}(s + \frac{n+1}{2} + l)$ and $\beta = \frac{1}{2}(s + \frac{n+1}{2} - l)$ and $\Gamma_n(s)$ is the Gamma function of degree n (see e. g. [2], p. 152).

We are now ready to state the analytic properties of $E_{l,0}^n$.

Theorem 4.5 [27, Theorem 3.16] *Let $E_{l,0}^n(\tau, s)$ be the Eisenstein series as in Definition 4.3 with respect to $\rho_{L,n}$. Then $E_{l,0}^n(\tau, s)$ has a meromorphic continuation in s to the whole complex plane. If L'/L is anisotropic and $|L'/L|$ odd, it satisfies the functional equation*

$$E_{l,0}^n(\tau, s - \frac{l}{2}) = \xi(2s - \frac{n+1}{2}, n, l, P) \prod_{p||L'/L|} \sum_{(\sigma,I,k)} \kappa_p(\sigma, I, k)\Delta_p(\sigma, q, k)$$

$$\times \prod_{j=1}^{k}(D_{p,j}(2s - \rho_n) - 1)E_{l,0}^n(\tau, \rho_n - s - \frac{l}{2}),$$
(4.26)

where σ runs over all elements in S_n with $\sigma^2 = $ id, $I_0 \cup \cdots \cup I_s$ over all σ-stable decompositions of I and k from zero to $s + 1$. In this formula we interpret any occurring sum or product with lower index bigger than the upper index to be zero or one, respectively.

As a corollary we phrase now the corresponding theorem for the case $n = 2$. To this end, we introduce some quantities, which comprise the contribution of the primes p dividing $|L'/L|$: We write $S_2 = \{\text{id}, \sigma\}$ and use the symbol I_0 in two ways. On the one hand, it means $I_0 = \{1, 2\}$ if the partition of I consists only of I_0. On the other hand, it means $I_0 = \{1\}$ if the partition of I is of the form $I = I_0 \cup I_1 = \{1\} \cup \{2\}$. Further, by $(L'/L)_p$ we mean the p-component of L'/L. For $\tau \in S_2$, any (τ-stable) decomposition I' of $I = \{1, 2\}$ and $k \in \{0, 1, 2\}$ we give explicit expressions for

$$K_p(\tau, I', k) = \kappa_p(\tau, I', k)\Delta_p(\tau, q, k) \prod_{j=1}^{k}(D_{p,j}(2s - \frac{3}{2}) - 1): \qquad (4.27)$$

$$K_p(\text{id}, I_0, 0) = \frac{(1 - p^{-1})^4}{1 - p^{-3}},$$

$$K_p(\text{id}, I_0 \cup I_1, 0) = \frac{p^{-1}(1 - p^{-1})^3}{1 - p^{-3}}, \qquad (4.28)$$

$$K_p(\sigma, I_0, 0) = \frac{p^{-2}(1 - p^{-1})^2}{1 - p^{-3}}.$$

For $k = 1$ and $k = 2$ we have to distinguish the cases $(L'/L)_p \cong (\mathbb{Z}/p\mathbb{Z})^2$ and $(L'/L)_p \cong \mathbb{Z}/p\mathbb{Z}$. In the latter case we have $K_p(\tau, I', k) = 0$ for all listed quantities in (4.29). Therefore, the following identities hold only for $(L'/L)_p \cong (\mathbb{Z}/p\mathbb{Z})^2$.

$$K_p(\text{id}, I_0, 1) = 2(1 - p^{-1})^4(\zeta_p(8s + 6) - 1),$$
$$K_p(\sigma, I_0, 1) = 2(1 - p^{-1})^2(\zeta_p(16s - 6) - 1),$$
$$K_p(\text{id}, I_0 \cup I_1, 1) = p(1 - p^{-1})^2((-1)\left(\frac{2}{p}\right) + \left(\frac{-1}{p}\right)^{(m-2)/2})(\zeta_p(8s - 6) - 1), \qquad (4.29)$$
$$K_p(\text{id}, I_0 \cup I_1, 2)$$
$$= 4p^2(1 - p^{-1})^3(1 + (-1)\left(\frac{-1}{p}\right)^{(m-2)/2}\left(\frac{2}{p}\right)p^{s + \frac{m}{2} - 2})(\zeta_p(4s - 4) - 1)(\zeta_p(8s - 6) - 1).$$

Corollary 4.6 *Let $E_{l,0}^2(\tau, s)$ be the Eisenstein series as in Definition 4.3 with respect to $\rho_{L,2}$. Then $E_{l,0}^2(\tau, s)$ has a meromorphic continuation in s to the whole complex plane. If L'/L is anisotropic and $|L'/L|$ odd, it satisfies the functional equation*

$$E_{l,0}^2(\tau, s - \frac{l}{2}) = \xi(2s - \frac{3}{2}, 2.l, P) \prod_{p||L'/L|} \left(C_p(\text{id}, I_0) + C_p(\text{id}, I_0 \cup I_1) + C_p(\sigma, I_0)\right)$$

$$\times E_{l,0}^2(\tau, \frac{3}{2} - s - \frac{l}{2}), \qquad (4.30)$$

Here

$$\xi(2 - \frac{3}{2}, 2.l, P) = \frac{(-1)^{l/2}4^{1-s}\pi^3}{|L'/L|} \frac{\Gamma_2(s)}{\Gamma_2(\alpha)\Gamma_2(\beta)} \frac{a_2^P(s)}{b_2^P(s)}$$

with the same meaning as above and

$$C_p(\mathrm{id}, I_0) = K_p(\mathrm{id}, I_0, 0) + K_p(\mathrm{id}, I_0, 1),$$

$$C_p(\mathrm{id}, I_0 \cup I_1) = K_p(\mathrm{id}, I_0 \cup I_1, 0) + K_p(\mathrm{id}, I_0 \cup I_1, 1) + K_p(\mathrm{id}, I_0 \cup I_1, 2),$$

$$C_p(\sigma, I_0) = K_p(\sigma, I_0, 0) + K_p(\sigma, I_0, 1).$$

Proof As indicated before, for the statement of Corollary 4.6 we have to calculate $K_p(\tau, I', k)$ in (4.27) for all $\tau \in S_2$, any τ-stable partition I' of $I = \{1, 2\}$ and all $k \in \{0, 1, 2\}$. Since $\Delta_p(\tau, q, 0) = 1$ and the product $\prod_{j=1}^{0}(D_{p,j}(2s - \frac{3}{2}) - 1)$ is empty and thus equal to 1, we have

$$K_p(\tau, I', 0) = \kappa_p(\tau, I', 0).$$

By the definition of $\Delta_p(\tau, q, 0)$, this identity holds for both possible cases of $(L'/L)_p$. The expressions for $K_p(\tau, I', 0)$ in (4.28) can be verified by a direct but tedious calculation. Again, the definition of $\Delta_p(\tau, q, k)$ for $k = 1, 2$ yields $K_p(\tau, I', k) = 0$ for all $\tau \in S_2$, all partitions I' of I and $(L'/L)_p \cong \mathbb{Z}/p\mathbb{Z}$. It remains to confirm the equations in (4.29). This is also a straightforward but tedious calculation. For all these computations we have to take the convention in Theorem 4.5 into account. □

Remark 4.7 Note that by Remark 3.18 in [27] the functional equation (4.30) remains valid for the Eisenstein series $E_{l,0}^{2*}$.

Since the case $n = 2$ lies mainly in our interest, we stick with it for the rest of this section and introduce now a vector-valued analogue of the Poincaré series in [2], Sect. 2.2, and a related Poincaré series. To this end, let $\mathcal{H} = \mathbb{H} \cup \mathbb{H}^-$, where $\mathbb{H}^-$ is the lower complex half-plane.

Definition 4.8 Let $d \in \mathbb{Z}$ be a positive integer, $D = \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})^+$, $l \in \mathbb{N}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{3-l}{2}$.

(i) $P_l : \mathbb{H} \times \mathbb{H} \to \mathbb{C}[L'/L] \otimes \mathbb{C}[L'/L]$ is defined by

$$P_l(\tau, \zeta, s) = \sum_{\lambda \in L'/L} \sum_{\gamma \in \Gamma_1} \det\left(\mathrm{Im}\begin{pmatrix} \tau & 0 \\ 0 & \zeta \end{pmatrix}\right)^s (\tau + \zeta)^{-l} |\tau + \zeta|^{-2s} \mathfrak{e}_\lambda \otimes \mathfrak{e}_\lambda \mid_{l,L} \gamma_{(\tau,1)},$$

(4.31)

where the subscript $(\tau, 1)$ means that γ acts on the first component and with respect to the variable τ. We further define a variant of P_l, which occurs in the course of the present paper. It already appeared in slightly less general scalar-valued form in Poincaré's work (see [16]). $P_l^+ : \mathcal{H} \times \mathcal{H} \to \mathbb{C}[L'/L] \otimes \mathbb{C}[L'/L]$ with

$$P_l^+(\tau, \zeta, s) = \sum_{\lambda \in L'/L} \sum_{\gamma \in \Gamma_1} \det\left(\mathrm{Im}\begin{pmatrix} \tau & 0 \\ 0 & \zeta \end{pmatrix}\right)^s (\tau - \zeta)^{-l} |\tau - \zeta|^{-2s} \mathfrak{e}_\lambda \otimes \mathfrak{e}_\lambda \mid_{l,L} \gamma_{(\tau,1)}.$$

(4.32)

(ii) Associated to P_l and its variant we define

$$\mathscr{P}_l : \mathbb{H} \times \mathbb{H} \to \mathbb{C}[L'/L] \otimes \mathbb{C}[L'/L], \quad \mathscr{P}_l(\tau, \zeta, D, s) = \sum_{M \in \Gamma_1 \backslash \Gamma_1 D \Gamma_1} P_l(\tau, \zeta, s) \mid_{l,L}^* M_{(\zeta,2)},$$

(4.33)

where the subscript $(\zeta, 2)$ indicates that M acts on the second component and with respect to the variable ζ and $\rho_{L,1}^*$. Corresponding to P_l^+ we put

$$\mathscr{P}_l^+ : \mathcal{H} \times \mathcal{H} \to \mathbb{C}[L'/L] \otimes \mathbb{C}[L'/L], \quad \mathscr{P}_l^+(\tau, \zeta, D, s)$$
$$= \sum_{M \in \Gamma_1 \backslash \Gamma_1 D \Gamma_1} P_l^+(\tau, \zeta, s) \mid_{l,L}^* M_{(\zeta,2)}. \tag{4.34}$$

Remark 4.9   (i)   The series P_l can be understood as vector-valued version of the Poincaré series $P_r^k(Z, W, g, s)$ introduced in [2], Sect. 2.2. It can also be interpreted as the non-holomorphic variant of the series $h_{\beta,n}$ defined in [22], Def. 5.1. In terms of the coefficients of $\rho_{L,1}$ the component function $P_l(\tau, \zeta, s)_\mu$ can be written as

$$\det \left(\mathrm{Im} \left(\begin{smallmatrix} \tau & 0 \\ 0 & \zeta \end{smallmatrix} \right) \right)^s \sum_{\gamma \in \Gamma_1} \left\langle \rho_{L,1}^{-1}(\gamma) \mathfrak{e}_\lambda, \mathfrak{e}_\mu \right\rangle j(\gamma, \tau)^{-l} |j(\gamma, \tau)|^{-2s} (\gamma \tau + \zeta)^{-l} |\gamma \tau + \zeta|^{-2s}. \tag{4.35}$$

Since $\rho_{L,1}$ factors through the finite group $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, the coefficient $\rho_{\lambda,\mu}^{-1}(\gamma)$ is bounded on Γ_1. Thus, for matters of convergence of (4.35) it suffices to study the scalar-valued Poincaré series

$$\det \left(\mathrm{Im} \left(\begin{smallmatrix} \tau & 0 \\ 0 & \zeta \end{smallmatrix} \right) \right)^s \sum_{\gamma \in \Gamma_1} j(\gamma, \tau)^{-l} |j(\gamma, \tau)|^{-2s} (\gamma \tau + \zeta)^{-l} |\gamma \tau + \zeta|^{-2s},$$

which is exactly the series $P_1^l(Z, W, g, s)$ in [2], Sect. 2.2, for $g \in \Gamma_1$. It follows from [2], Sec. 2.2, that each component function of $P_l(\tau, \zeta, s)_\mu$ is absolutely and uniformly convergent for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{3-l}{2}$ on any product $V_1(\delta) \times V_1(\delta)$, where

$$V_1(\delta) = \{ z = x + iy \in \mathbb{H} \mid y \geq \delta \text{ and } x^2 \leq \frac{1}{\delta} \}$$

with $\delta > 0$, and thereby represents a real-analytic function on $\mathbb{H}^2$. Thus, the usual argument shows that P_l transforms under the action of Γ_1 with respect to the variable τ like a vector-valued modular form of weight l and type $\rho_{L,1}$.

Since the sum $\sum_{M \in \Gamma_1 \backslash \Gamma_1 D \Gamma_1}$ is finite, the same holds for the Poincaré series $\mathscr{P}_l$. In terms of the function $\varphi_{l,s}$ (see (2.9)), we may write $\mathscr{P}_l$ in the following more explicit form

$$\sum_{\lambda \in L'/L} \sum_{M \in \Gamma_1 \backslash \Gamma_1 D \Gamma_1}$$
$$\times \sum_{\gamma \in \Gamma_1} \det \left(\mathrm{Im} \left(\begin{smallmatrix} \tau & 0 \\ 0 & \zeta \end{smallmatrix} \right) \right)^s \varphi_{l,s} \left(j(\gamma, \tau) j(M, \zeta) (\gamma \tau + M \zeta) \right) \rho_{L,1}^{-1}(\gamma) \mathfrak{e}_\lambda \otimes \rho_{L,1}^{*-1}(M) \mathfrak{e}_\lambda. \tag{4.36}$$

(ii)   The series P_l^+ has similar properties as P_l. More specifically, it converges absolutely and uniformly on compact subsets of $\mathcal{H} \times \mathcal{H}$.

It is thereby real-analytic on $\mathcal{H} \times \mathcal{H}$, leaving out possible poles. Indeed, if τ, ζ are both in the upper or lower half-plane, P_l^+ has a pole at $\tau = \zeta$. In all other cases no poles can occur. Also, the usual argument shows that P_l^+ transforms with respect to $\tau \in \mathbb{H}$ like a vector-valued modular form.

If τ is an element of $\mathbb{H}$, the aforementioned statements can be deduced from Theorem 3A and Section 3C in Section V of [16].

If $\tau \in \mathbb{H}^-$ and $\zeta \in \mathbb{H}$, P_l^+ inherits the analytic properties of P_l. To be specific, by employing the same argument as in (i), we may estimate each of the component

functions of P_l^+ by

$$\sum_{\gamma \in \Gamma_1} |j(\gamma, \tau)|^{-2s-l} |(\gamma \tau - \zeta)|^{-l-2s} = \sum_{\gamma \in \Gamma_1} |j(\gamma, -z)|^{-2s-l} |(\gamma(-z) - \zeta)|^{-l-2s}$$

$$= \sum_{\gamma \in \Gamma_1} |j(\widetilde{\gamma}, z)|^{-2s-l} |(\widetilde{\gamma} z + \zeta)|^{-l-2s},$$

where we replaced $\tau \in \mathbb{H}^-$ with $-z$, $z \in \mathbb{H}$, and used the fact that $\gamma(-z) = -(\widetilde{\gamma} z)$. The latter sum is up to a constant nothing else but an estimate of the series (4.35). If τ, ζ are both in $\mathbb{H}^-$, the same reasoning as before shows that P_l^+ is bounded by

$$\sum_{\gamma \in \Gamma_1} |j(\widetilde{\gamma}, z)|^{-2s-l} |(\widetilde{\gamma} z + \zeta)|^{-l-2s}$$

with $z \in \mathbb{H}$ and $\zeta \in \mathbb{H}^-$. This is up to a constant an estimate of $P_{l|\mathbb{H} \times \mathbb{H}^-}^+$. Again, in all considered cases the Poincaré series $\mathscr{P}_l^+$ shares the same properties as P_l^+.

The following theorem provides some further properties of P_l^+ and $\mathscr{P}_l^+$, which will be vital later on.

Theorem 4.10 *Let $l \in \mathbb{N}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{3-l}{2}$, $f \in S_l(\rho_{L,1})$ and*

$$C(l, s) = (-1)^{\frac{l}{2}} 2^{2-2s-l+1} \pi \frac{\Gamma(l+s-1)}{\Gamma(l+s)}. \tag{4.37}$$

(i) *Then we have*

$$\sum_{\lambda \in L'/L} \left(\int_{\Gamma_1 \backslash \mathbb{H}} \left\langle f(\tau) \otimes \mathfrak{e}_\lambda, P_l^+(\tau, \overline{\zeta}, \overline{s}) \right\rangle_2 \mathrm{Im}(\tau)^l d\mu(\tau) \right) \mathfrak{e}_\lambda = (-1)^{-s} C(l, s) f(\zeta) \tag{4.38}$$

for all $\zeta \in \mathbb{H}$.

(ii) *Let $D = \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})^+$ and $T(D)$ the Hecke operator defined in Sect. 4.1.1. Then*

$$\sum_{\lambda \in L'/L} \left(\int_{\Gamma_1 \backslash \mathbb{H}} \left\langle f(\tau) \otimes \mathfrak{e}_\lambda, \mathscr{P}_l^+(\tau, \overline{\zeta}, D, \overline{s}) \right\rangle_2 \mathrm{Im}(\tau)^l d\mu(\tau) \right) \mathfrak{e}_\lambda$$

$$= (-1)^{-s} C(l, s) \left(f \mid_{l,L} T(D) \right)(\zeta) \tag{4.39}$$

for all $\zeta \in \mathbb{H}$.

Proof (*i*): It can easily be confirmed by direct calculation using the relation (3.12) that

$$\left\langle f(\tau) \otimes \mathfrak{e}_\lambda, P_l(\tau, \overline{\zeta}, s) \right\rangle_2$$
$$= \left\langle f(\tau), \sum_{\mu \in L'/L} \sum_{\gamma \in \Gamma_1} \det \left(\mathrm{Im} \begin{pmatrix} \tau & 0 \\ 0 & \zeta \end{pmatrix} \right)^{\overline{s}} (\tau - \overline{\zeta})^{-l} |\tau - \overline{\zeta}|^{-2\overline{s}} \mathfrak{e}_\mu \mid_{l,L} \gamma \right\rangle_1 \left\langle \mathfrak{e}_\lambda \otimes \mathfrak{e}_\mu \right\rangle_1. \tag{4.40}$$

The last expression shows that the integral in (4.38) as Petersson scalar product $(\cdot,\cdot)_1$ is well-defined since f is a cusp form and P_l^+ transforms like a modular form in $M_l(\rho_{L,1})$ with respect to τ. Using (4.40), the left-hand side of (4.38) becomes

$$\sum_{\lambda \in L'/L} \left(\int_{\Gamma_1 \backslash \mathbb{H}} \left\langle f(\tau), \sum_{\gamma \in \Gamma_1} \det\left(\text{Im}\begin{pmatrix} \tau & 0 \\ 0 & \zeta \end{pmatrix}\right)^{\bar{s}} (\tau - \overline{\zeta})^{-l} |\tau - \overline{\zeta}|^{-2\bar{s}} \mathfrak{e}_\lambda \mid_{l,L} \gamma \right\rangle_1 \text{Im}(\tau)^l d\mu(\tau) \right) \mathfrak{e}_\lambda,$$
(4.41)

which can be seen as non-holomorphic version of the kernel operator K_1 defined in [22], Definition 5.3. To prove the stated assertion, we may use the same unfolding trick as laid out in the proof of Theorem 5.6 in [22]. In doing so, we obtain

$$(-1)^{-s} \text{Im}(\zeta)^s \int_{\Gamma_1 \backslash \mathbb{H}} \left\langle f(\tau), \sum_{\gamma \in \Gamma_1} \varphi_{l,\bar{s}}\left(j(\gamma,\tau)(\tau - \overline{\zeta})\right) \rho_{L,1}^{-1}(\gamma)\mathfrak{e}_\lambda \right\rangle_1 \text{Im}(\tau)^{l+s} d\mu(\tau)$$
$$= (-1)^{-s} \text{Im}(\zeta)^s \int_{\mathbb{H}} \left\langle f(\tau), (\tau - \overline{\zeta})^{-l} |\tau - \overline{\zeta}|^{-2\bar{s}} \mathfrak{e}_\lambda \right\rangle_1 \text{Im}(\tau)^{l+s} d\mu(\tau).$$
(4.42)

Since each component of f is a cusp form for $\Gamma(N)$, the latter integral in (4.42) can be evaluated explicitly by means of the proposition in [2], Sect. 2.2:

$$(-1)^{-s} \text{Im}(\zeta)^s \int_{\mathbb{H}} f_\lambda(\tau)(\overline{\tau} - \zeta)^{-l} |\tau - \overline{\zeta}|^{-2s} \text{Im}(\tau)^{l+s} d\mu(\tau) = (-1)^{-s} C(l,s) f_\lambda(\zeta)$$
(4.43)

for any $\zeta \in \mathbb{H}$.

(ii): Similar to (i) we have

$$\left\langle f(\tau) \otimes \mathfrak{e}_\lambda, \mathscr{P}_l^+(\tau, \overline{\zeta}, D, \bar{s}) \right\rangle_2 = \sum_{M \in \Gamma_1 \backslash \Gamma_1 D \Gamma_1} \sum_{\mu \in L'/L} \sum_{\gamma \in \Gamma_1} \times$$
$$\left\langle f(\tau), \det\left(\text{Im}\begin{pmatrix} \tau & 0 \\ 0 & \zeta \end{pmatrix}\right)^{\bar{s}} \varphi_{l,\bar{s}}\left(j(\gamma,\tau)j(M,\overline{\zeta})(\gamma\tau - M\overline{\zeta})\right) \rho_{L,1}^{-1}(\gamma)\mathfrak{e}_\mu \right\rangle_1 \left\langle \mathfrak{e}_\lambda, \rho_{L,1}^{*-1}(M)\mathfrak{e}_\mu \right\rangle_1.$$
(4.44)

Thus, as before,

$$\int_{\Gamma_1 \backslash \mathbb{H}} \left\langle f(\tau) \otimes \mathfrak{e}_\lambda, \mathscr{P}_l^+(\tau, \overline{\zeta}, D, \bar{s}) \right\rangle_2 \text{Im}(\tau)^l d\mu(\tau)$$
$$= (-1)^{-s} \sum_{M \in \Gamma_1(D) \backslash \Gamma_1} \text{Im}(M\zeta)^s j(M,\zeta)^{-l} \sum_{\mu \in L'/L} \left\langle \mathfrak{e}_\lambda, \rho_{L,1}^{*-1}(M)\mathfrak{e}_\mu \right\rangle_1$$
$$\times \int_{\Gamma_1 \backslash \mathbb{H}} \left\langle f(\tau), \sum_{\gamma \in \Gamma_1} \varphi_{l,\bar{s}}\left(j(\gamma,\tau)(\gamma\tau - M\overline{\zeta})\right) \rho_{L,1}^{-1}(\gamma)\mathfrak{e}_\mu \right\rangle_1 \text{Im}(\tau)^{l+s} d\mu(\tau).$$
(4.45)

Unfolding the latter integral and applying (4.43) subsequently, yields

$$\sum_{\lambda \in L'/L} \left(\int_{\Gamma_1 \backslash \mathbb{H}} \left\langle f(\tau) \otimes \mathfrak{e}_\lambda, \mathscr{P}_l^+(\tau, \overline{\zeta}, D, \bar{s}) \right\rangle_2 \text{Im}(\tau)^l d\mu(\tau) \right) \mathfrak{e}_\lambda$$
$$= (-1)^{-s} C(l,s) \sum_{M \in \Gamma_1 \backslash \Gamma_1 D \Gamma_1} \sum_{\mu \in L'/L} j(M,\zeta)^{-l} f_\mu(M\zeta) \sum_{\lambda \in L'/L} \left\langle \mathfrak{e}_\lambda, \rho_{L,1}^{*-1}(M)\mathfrak{e}_\mu \right\rangle_1 \mathfrak{e}_\lambda.$$
(4.46)

The right-hand side of (4.46) is nothing else than $(-1)^{-s} C(l,s)(f \mid_{l,L} T(D))(\zeta)$.   □

5 Garrett–Böcherer decomposition of vector-valued Siegel Eisenstein series

In this section we present a decomposition of the Siegel Eisenstein series $E_{l,0}^2$ in terms of $E_{l,0}^1$ and the Poincaré series $\mathscr{P}_l^+$. Such a decomposition was developed by Garrett [13] and Böcherer [2] for scalar-valued holomorphic and non-holomorphic Siegel Eisenstein series, respectively. It is based on an explicit system of representatives of $\Gamma_{n+m,0}\backslash\Gamma_{n+m}/\iota_{n+m}(\Gamma_m \times \Gamma_n)$ determined by Garrett (see (2.5) for the definition of $\iota_{n,m}$). Since we are dealing with the same groups for $n = m = 1$, we may use Garrett's results. They become considerably easier in this special case, so we summarize them here in the following two theorems. Subsequently, we will use these theorems to state and prove our vector-valued version of the Garrett–Böcherer decomposition.

Theorem 5.1 *A complete set of representatives for the double coset $\Gamma_{2,0}\backslash\Gamma_2/\iota_{1,1}(\Gamma_1 \times \Gamma_1)$ is given by*

$$\mathscr{M}_0 \cup \mathscr{M}_1,$$

where

$$\mathscr{M}_0 = \{U_2(0_2)\} \text{ and } \mathscr{M}_1 = \left\{U_2 \left(\begin{smallmatrix} 0 & d \\ d & 0 \end{smallmatrix}\right) \mid d \in \mathbb{N}\right\}. \tag{5.1}$$

Let $g_d \in \mathscr{M}_i$, $d \in \mathbb{N}_0$. Tailored to the case $n = m = 1$ the theorem in [13], §3, specifies a complete set of left coset representatives of $\Gamma_{2,0}\backslash\Gamma_{2,0}\, g_d\, l_{1,1}(\Gamma_1 \times \Gamma_1)$.

Theorem 5.2 *Let $d \in \mathbb{N}_0$, $r \in \{0, 1\}$ and $g_d \in \mathscr{M}_r$. Then a complete set of representatives of the $\Gamma_{2,0}$-left cosets in $\Gamma_{2,0}\, g_d\, l_{1,1}(\Gamma_1 \times \Gamma_1)$ is given by*

$$\left\{g_d\, (\gamma g)^{\uparrow}(Mh)^{\downarrow} \mid \gamma \in \Gamma_r,\, g \in \Gamma_{1,r}\backslash\Gamma_1,\, M \in \Gamma_r \left(\begin{smallmatrix} 0 & d^{-1} \\ d & 0 \end{smallmatrix}\right)\backslash\Gamma_r,\, h \in \Gamma_{1,r}\backslash\Gamma_1\right\}. \tag{5.2}$$

Note that we use the convention that $\Gamma_0 = \{1_2\}$ and accordingly $\Gamma_0 \left(\begin{smallmatrix} 0 & d \\ d^{-1} & 0 \end{smallmatrix}\right)\backslash\Gamma_0 = \{1_2\}$. We also have that $\Gamma_{1,1}\backslash\Gamma_1 = \{1_2\}$.

Based on these theorems a vector-valued variant of the pullback formula in [13, Sect. §5], and [2], involving the Siegel Eisenstein series (4.17), can be given.

Theorem 5.3 *Let $d \in \mathbb{Z}$ be a positive integer, $D = \left(\begin{smallmatrix} d & 0 \\ 0 & d^{-1} \end{smallmatrix}\right)$ and $E_{l,0}^{2*}$ be defined as in Definition 4.3. Then for all $\tau, \zeta \in \mathbb{H}$*

$$
\begin{aligned}
&E_{l,0}^{2*}\left(\left(\begin{smallmatrix} \tau & 0 \\ 0 & \zeta \end{smallmatrix}\right), s\right) \\
&= E_{l,0}^{1*}(\tau, s) \otimes E_{l,0}^{1*}(\zeta, s) + \frac{e(\mathrm{sig}(L)/8)}{|L'/L|^{1/2}} \sum_{d \geq 1} \frac{g_d(L)}{g(L)} d^{-l-2s} \mathscr{P}_l^+(-\tau, \zeta, D, s).
\end{aligned}
\tag{5.3}
$$

Remark 5.4 Since $E_{l,0}^2$ is absolutely and uniformly convergent on compact subsets of $\mathbb{H}^2$, the same holds for any subseries occurring on the right-hand side of (5.3). In particular, $\sum_{d\in\mathbb{N}} \frac{g_d(L)}{g(L)} d^{-l-2s} \mathscr{P}_l^+(-\tau, \zeta, D, s)$ is normally convergent for $\mathrm{Re}(s) > \frac{3-l}{2}$.

Proof of Theorem 5.3: The proof is an adaption of the one given in [13]. All steps concerning the factor of automorphy j_2 carry over immediately. The parts in $E_{l,0}^{2*}$ coming from the Weil representation $\rho_{L,2}^*$ have to be treated separately.

According to Theorems 5.1 and 5.2 we have

$$
\begin{aligned}
E_{l,0}^{2*}\left(\left(\begin{smallmatrix}\tau & 0\\ 0 & \zeta\end{smallmatrix}\right), s\right) =& \sum_{g\in\Gamma_{1,0}\backslash\Gamma_1}\sum_{h\in\Gamma_{1,0}\backslash\Gamma_1} \det\left(\operatorname{Im}\left(\begin{smallmatrix}\tau & 0\\ 0 & \zeta\end{smallmatrix}\right)\right)^s \mathfrak{e}_0 \mid_{l,L}^{2*} g^{\uparrow} h^{\downarrow} \\
&+ \sum_{d\in\mathbb{N}}\sum_{M\in\Gamma_1(d)\backslash\Gamma_1}\sum_{\gamma\in\Gamma_1} \det\left(\operatorname{Im}\left(\begin{smallmatrix}\tau & 0\\ 0 & \zeta\end{smallmatrix}\right)\right)^s \mathfrak{e}_0 \mid_{l,L}^{2*} U_2\left(\begin{smallmatrix}0 & d\\ d & 0\end{smallmatrix}\right)\gamma^{\uparrow} M^{\downarrow}.
\end{aligned}
\tag{5.4}
$$

Note that we have replaced the left cosets over $\Gamma_1\left(\begin{smallmatrix}0 & d^{-1}\\ d & 0\end{smallmatrix}\right)\backslash\Gamma_1$ with $\Gamma_1(d)\backslash\Gamma_1$. This amounts to replace M with $\ell(M)$ (see Sect. 2 for the notation), which only alters the order of summation but not the whole expression. We consider both of the summands above separately.

Starting with the first summand, note that a by direct computation we find

$$
j_2\left(g^{\uparrow} h^{\downarrow}, \left(\begin{smallmatrix}\tau & 0\\ 0 & \zeta\end{smallmatrix}\right)\right) = j(g,\tau)j(h,\zeta).
\tag{5.5}
$$

For the part involving the Weil representation $\rho_{L,2}^*$ we exploit (3.19) and (3.20) and infer that

$$
\begin{aligned}
\rho_{L,2}^{*-1}(g^{\uparrow} h^{\downarrow})\mathfrak{e}_{(0,0)} &= \rho_{L,2}^{*-1}(g^{\uparrow})\left(\mathfrak{e}_0 \otimes \rho_{L,1}^{*-1}(h)\mathfrak{e}_0\right) \\
&= \rho_{L,1}^{*-1}(g)\mathfrak{e}_0 \otimes \rho_{L,1}^{*-1}(h)\mathfrak{e}_0.
\end{aligned}
\tag{5.6}
$$

If we insert the right-hand side of (5.5) and (5.6), we may write in terms of (2.9)

$$
\begin{aligned}
&\sum_{g\in\Gamma_\infty\backslash\Gamma_1}\sum_{h\in\Gamma_\infty\backslash\Gamma_1} \det\left(\operatorname{Im}\left(\begin{smallmatrix}\tau & 0\\ 0 & \zeta\end{smallmatrix}\right)\right)^s \mathfrak{e}_0 \otimes \mathfrak{e}_0 \mid_{l,L}^{2*} g^{\uparrow} h^{\downarrow} \\
&= \sum_{g\in\Gamma_\infty\backslash\Gamma_1}\sum_{h\in\Gamma_\infty\backslash\Gamma_1} \det\left(\operatorname{Im}\left(\begin{smallmatrix}\tau & 0\\ 0 & \zeta\end{smallmatrix}\right)\right)^s \varphi_{l,s}\left(j(g,\tau)j(h,\zeta)\right)\rho_{L,1}^{*-1}(g)\mathfrak{e}_0 \otimes \rho_L^{*-1}(h)\mathfrak{e}_0 \\
&= E_{l,0}^{1*}(\tau,s) \otimes E_{l,0}^{1*}(\zeta,s).
\end{aligned}
$$

For the second summand of (5.4), a straightforward calculation, using the cocycle relation of j_2 twice, yields

$$
j_2\left(\left(U_2\left(\begin{smallmatrix}0 & d\\ d & 0\end{smallmatrix}\right)\gamma^{\uparrow} M^{\downarrow}, \left(\begin{smallmatrix}\tau & 0\\ 0 & \zeta\end{smallmatrix}\right)\right)\right) = \left(1 - (\gamma\tau)\cdot d^2(M\zeta)\right)j(\gamma,\tau)j(M,\zeta).
$$

On the other hand, by means of Lemma 3.4 and 3.5 we have

$$
\begin{aligned}
\rho_{L,2}^{*-1}\left(U_2\left(\begin{smallmatrix}0 & d\\ d & 0\end{smallmatrix}\right)\gamma^{\uparrow} M^{\downarrow}\right)\mathfrak{e}_0 \otimes \mathfrak{e}_0 &= \frac{1}{|L'/L|}\sum_{\mu,\nu\in L'/L} e\left(-(\mu,\nu)\right)\rho_{L,2}^{*-1}(M^{\downarrow})\rho_{L,2}^{*-1}(\gamma^{\uparrow})\mathfrak{e}_{d\mu}\otimes\mathfrak{e}_\nu \\
&= \frac{1}{|L'/L|}\sum_{\mu,\nu\in L'/L} e\left(-(\mu,\nu)\right)\rho_{L,2}^{*-1}(M^{\downarrow})(\rho_{L,1}^{*-1}(\gamma)\mathfrak{e}_{d\mu}\otimes\mathfrak{e}_\nu) \\
&= \frac{1}{|L'/L|}\sum_{\mu,\nu\in L'/L} e\left(-(\mu,\nu)\right)\rho_{L,1}^{*-1}(\gamma)\mathfrak{e}_{d\mu}\otimes\rho_{L,1}^{*-1}(M)\mathfrak{e}_\nu.
\end{aligned}
\tag{5.7}
$$

The transformation $\gamma \mapsto S^{-1}\widetilde{\gamma}$ (see (3.9) for the definition of $\widetilde{\gamma}$) leaves the sum over $\gamma \in \Gamma_1$ and therefore the latter summand of (5.4) invariant. The subsequent calculations can be easily verified

$$
\begin{aligned}
j_2\left(U_2\left(\begin{smallmatrix} 0 & d \\ d & 0 \end{smallmatrix}\right)(S\widetilde{\gamma})^{\uparrow}M^{\downarrow},\left(\begin{smallmatrix} \tau & 0 \\ 0 & \zeta \end{smallmatrix}\right)\right) &= \left(1 - (S^{-1}\widetilde{\gamma}\tau)\cdot d^2(M\zeta)\right)j(S^{-1}\gamma,\tau)j(M,\zeta) \\
&= \left(\gamma(-\tau) - d^2 M\zeta\right)j(\gamma,-\tau)j(M,\zeta) \\
&= d\left(\gamma(-\tau) - \left(\left(\begin{smallmatrix} d & 0 \\ 0 & d^{-1} \end{smallmatrix}\right)M\right)\zeta\right)j(\gamma,-\tau)j\left(\left(\begin{smallmatrix} d & 0 \\ 0 & d^{-1} \end{smallmatrix}\right)M,\zeta\right).
\end{aligned}
$$

Taking (5.7) and (3.6) into account, the corresponding calculations on the level of the Weil representation yield

$$
\begin{aligned}
&\rho_{L,2}^{*-1}\left(U_2\left(\begin{smallmatrix} 0 & d \\ d & 0 \end{smallmatrix}\right)\gamma^{\uparrow}M^{\downarrow}\right)\mathfrak{e}_0\otimes\mathfrak{e}_0 \\
&= \frac{1}{|L'/L|}\sum_{\mu,\nu\in L'/L}e\left(-(\mu,\nu)\right)\rho_{L,1}^{*-1}(\widetilde{\gamma})\rho_{L,1}^{*}(S)\mathfrak{e}_{d\mu}\otimes\rho_{L,1}^{*-1}(M)\mathfrak{e}_{\nu} \\
&= \frac{e(\mathrm{sig}(L)/8)}{|L'/L|^{3/2}}\sum_{\varrho,\nu\in L'/L}\sum_{\mu\in L'/L}e\left((\mu,d\varrho-\nu)\right)\rho_{L,1}^{-1}(\gamma)\mathfrak{e}_{\varrho}\otimes\rho_{L,1}^{*-1}(M)\mathfrak{e}_{\nu} \\
&= \frac{e(\mathrm{sig}(L)/8)}{|L'/L|^{1/2}}\sum_{\varrho\in L'/L}\rho_{L,1}^{-1}(\gamma)\mathfrak{e}_{\varrho}\otimes\rho_{L,1}^{*-1}(M)\mathfrak{e}_{d\varrho}.
\end{aligned}
$$

For the second equation we have additionally used (3.10).

With (4.13) in mind we may then rewrite the last expression in the form

$$
\frac{e(\mathrm{sig}(L)/8)}{|L'/L|^{1/2}}\frac{g_d(L)}{g(L)}\sum_{\varrho\in L'/L}\rho_{L,1}^{-1}(\gamma)\mathfrak{e}_{\varrho}\otimes\rho_{L,1}^{*-1}\left(\left(\begin{smallmatrix} d & 0 \\ 0 & d^{-1} \end{smallmatrix}\right)M\right)\mathfrak{e}_{\varrho}. \tag{5.8}
$$

Putting together the rearrangements above, we obtain

$$
\begin{aligned}
&\sum_{d\in\mathbb{N}}\sum_{M\in\Gamma_1(d)\backslash\Gamma_1}\sum_{\gamma\in\Gamma_1}\det\left(\mathrm{Im}\left(\begin{smallmatrix} \tau & 0 \\ 0 & \zeta \end{smallmatrix}\right)\right)^s\mathfrak{e}_0\mid_{l,L}^2 U_2\left(\begin{smallmatrix} 0 & d \\ d & 0 \end{smallmatrix}\right)\gamma^{\uparrow}M^{\downarrow} \\
&= \sum_{d\in\mathbb{N}}\sum_{R\in\Gamma_1\backslash\Gamma_1\left(\begin{smallmatrix} d & 0 \\ 0 & d^{-1} \end{smallmatrix}\right)\Gamma_1}\sum_{\gamma\in\Gamma_1}\det\left(\mathrm{Im}\left(\begin{smallmatrix} \tau & 0 \\ 0 & \zeta \end{smallmatrix}\right)\right)^s d^{-l-2s}\varphi_{l,s}\left(j(\gamma,-\tau)j(R,\zeta)(\gamma(-\tau)-R\zeta)\right) \\
&\quad\times\frac{e(\mathrm{sig}(L)/8)}{|L'/L|^{1/2}}\frac{g_d(L)}{g(L)}\sum_{\varrho\in L'/L}\rho_{L,1}^{-1}(\gamma)\mathfrak{e}_{\varrho}\otimes\rho_{L,1}^{*-1}(R)\mathfrak{e}_{\varrho}.
\end{aligned}
$$

$\square$

6 Standard zeta function of an eigenform

In [9], p. 251, it was proposed to associate a zeta function

$$
\sum_{\substack{d\in\mathbb{N}\\(d,N)=1}}\lambda_d(f)d^{-s}
$$

to a common eigenform f of all Hecke operators $T\left(\begin{smallmatrix} d^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $(d,N)=1$, where

$$
f\mid_{l,L}T\left(\begin{smallmatrix} d^2 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \lambda_d(f)f.
$$

In this section we study the analytic properties of a slightly different zeta function. Attached to an eigenform f of *all* Hecke operators $T\left(\begin{smallmatrix} d^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we define

$$Z(s,f) = \sum_{d \in \mathbb{N}} \lambda_d(f) d^{-s}. \tag{6.1}$$

In consistency with [2] the zeta function $Z(s,f)$ can be viewed as the standard zeta function of f.

Remark 6.1   (i) As was already stated in [9], $Z(s,f)$ converges for Re(s) sufficiently large. We obtain this result as a by-product of the subsequent studies of the analytic properties of $Z(s,f)$.

 (ii) Under certain assumptions on the discriminant form one can prove the existence of a common eigenform of all Hecke operators $T\left(\begin{smallmatrix} d^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $d \in \mathbb{N}$. This is possible if a multiplicity one theorem for $S_l(\rho_{L,1})$ holds. In [30], Theorem 41, conditions for the validity of such a theorem are stated. These conditions depend heavily on the decomposition of the Weil representation $\rho_{L,1}$ into irreducible subrepresentations. The multiplicities of these subrepresentations encode the dimension of the common eigenspace for a set of eigenvalues λ_d for all Hecke operators $T\left(\begin{smallmatrix} d^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $(d,N) = 1$. If all occurring irreducible subrepresentations have multiplicity one, the same holds for the before mentioned dimension of common eigenspaces. The decomposition of $\rho_{L,1}$ into irreducible subrepresentations is well known, see e.g. [19], [20]. Among other things, it depends on the structure of the discriminant form L'/L. If for example each p-group of L'/L consists of a single Jordan block of the form $(\mathbb{Z}/p^\lambda\mathbb{Z}, \frac{rx^2}{p^\lambda})$, $\rho_{L,1}$ decomposes into irreducible subrepresentations of multiplicity one.

6.1 Analytic properties of *Z*(*s*, *f*)

In this section we will prove that $Z(s,f)$ can be continued meromorphically to the whole complex s-plane. Also, we are able to establish a functional equation of $Z(s,f)$ under the assumption that the discriminant form L'/L is anisotropic. The proof is an adaption of the corresponding result in [2] to the vector-valued setting. A first step is the following theorem.

Theorem 6.2 *Let* $f \in S_l(\rho_{L,1})$ *be a cusp form with Fourier expansion*

$$f(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\mu) \\ n>0}} a(\mu,n) e(n\tau).$$

Then for $\frac{l}{2} + \mathrm{Re}(s) > 1$ *we have*

$$\sum_{\lambda \in L'/L} \left(\int_{\Gamma_1 \backslash \mathbb{H}} \left\langle f(\tau) \otimes \mathfrak{e}_\lambda, E_{l,0}^{1*}(-\tau,\overline{s}) \otimes E_{l,0}^{1*}(\overline{\zeta},\overline{s}) \right\rangle_2 \mathrm{Im}(\tau)^l d\mu(\tau) \right) \mathfrak{e}_\lambda = 0. \tag{6.2}$$

Here $E_{l,0}^{1*}(-\tau,\overline{s})$ *and* $E_{l,0}^{1*}(\overline{\zeta},\overline{s})$ *are defined by* (4.19).

Proof Similar to (4.40) and (4.44) a straightforward calculation using the relation (3.12) gives

$$\left\langle f(\tau) \otimes \mathfrak{e}_\lambda, E_{l,0}^{1*}(-\tau,\overline{s}) \otimes E_{l,0}^{1*}(\overline{\zeta},\overline{s}) \right\rangle_2 = (-1)^{2s} \left\langle f(\tau) \otimes \mathfrak{e}_\lambda, E_{l,0}^1(\tau,\overline{s}) \otimes E_{l,0}^1(-\overline{\zeta},\overline{s}) \right\rangle_2$$
$$= (-1)^{2s} \left\langle f(\tau), E_{l,0}^1(\tau,\overline{s}) \right\rangle_1 \left\langle \mathfrak{e}_\lambda, E_{l,0}^1(-\overline{\zeta},\overline{s}) \right\rangle_1.$$

Replacing this with the integrand in (6.2), we find that the integral in (6.2) is equal to

$$(-1)^{2s} \left(\sum_{\lambda \in L'/L} \langle \mathfrak{e}_\lambda, E^1_{l,0}(-\overline{\zeta}, \overline{s}) \rangle_1 \mathfrak{e}_\lambda \right) \int_{\Gamma_1 \backslash \mathbb{H}} \langle f(\tau), E^1_{l,0}(\tau, \overline{s}) \rangle_1 \operatorname{Im}(\tau)^l d\mu(\tau). \tag{6.3}$$

The integral in (6.3) is the Petesson scalar product of f and $E^1_{l,0}$. As such, it is well-defined. For its evaluation we adapt the calculations in the proof of [5], Proposition 1.5, to our situation. The usual unfolding argument (see the proof of Thm. 5.6 in [22]) yields

$$\int_{\Gamma_1 \backslash \mathbb{H}} \langle f(\tau), E^1_{l,0}(\tau, \overline{s}) \rangle_1 \operatorname{Im}(\tau)^l d\mu(\tau) = \int_{\Gamma_1 \backslash \mathbb{H}} \sum_{g \in \Gamma_\infty \backslash \Gamma_1} \langle f(g\tau), \mathfrak{e}_0 \rangle_1 \operatorname{Im}(g\tau)^{l+s} d\mu(\tau)$$

$$= \int_{\Gamma_\infty \backslash \mathbb{H}} \langle f(\tau), \mathfrak{e}_0 \rangle_1 \operatorname{Im}(\tau)^{l+s-2} dx dy.$$

Inserting the Fourier expansion of f_0 into the last integral, we find

$$\int_0^\infty \int_0^1 \sum_{\substack{n \in \mathbb{Z} \\ n > 0}} a(0, n) e(n\tau) \operatorname{Im}(\tau)^{l+s-2} dx dy$$

$$= \int_0^\infty \sum_{\substack{n \in \mathbb{Z} \\ n > 0}} a(0, n) e^{2\pi n y} y^{l+s-2} dy \left(\int_0^1 e^{2\pi i n x} dx \right)$$

$$= 0.$$

$\square$

In order to study the analytic properties of the standard zeta function, we express it basically as a Petersson scalar product of f with the restricted Eisenstein series $E^2_{l,0}(\left(\begin{smallmatrix} \tau & 0 \\ 0 & \zeta \end{smallmatrix} \right), s)$. This approach is well known and has been applied in several settings, see e. g. [2], [1] or [4].

Let $f \in S_l(\rho_{L,1})$ be a cusp form and $E^2_{l,0}$ be the Eisenstein series as in Definition 4.3. For $\zeta \in \mathbb{H}$ and $l + 2 \operatorname{Re}(s) > 3$ we form a vector-valued version of a Rankin–Selberg type integral

$$\sum_{\lambda \in L'/L} \left(\int_{\Gamma_1 \backslash \mathbb{H}} \left\langle f(\tau) \otimes \mathfrak{e}_\lambda, E^2_{l,0}(\left(\begin{smallmatrix} \tau & 0 \\ 0 & -\overline{\zeta} \end{smallmatrix} \right), \overline{s}) \right\rangle_2 \operatorname{Im}(\tau)^l d\mu(\tau) \right) \mathfrak{e}_\lambda. \tag{6.4}$$

Remark 6.3 In view of Theorem 5.3 we may write the integral in (6.4) in the form

$$\int_{\Gamma_1 \backslash \mathbb{H}} \left\langle f(\tau) \otimes \mathfrak{e}_\lambda, E^{2*}_{l,0}(\left(\begin{smallmatrix} -\tau & 0 \\ 0 & \overline{\zeta} \end{smallmatrix} \right), \overline{s}) \right\rangle_2 \operatorname{Im}(\tau)^l d\mu(\tau)$$

$$= \int_{\Gamma_1 \backslash \mathbb{H}} \left\langle f(\tau) \otimes \mathfrak{e}_\lambda, E^{1*}_{l,0}(-\tau, \overline{s}) \otimes E^{1*}_{l,0}(\overline{\zeta}, \overline{s}) \right\rangle_2 \operatorname{Im}(\tau)^l d\mu(\tau) \tag{6.5}$$

$$+ \frac{e(\operatorname{sig}(L)/8)}{|L'/L|^{1/2}} \sum_{d \in \mathbb{N}} \frac{g_d(L)}{g(L)} d^{-l-2s} \int_{\Gamma_1 \backslash \mathbb{H}} \left\langle f(\tau) \otimes \mathfrak{e}_\lambda, \mathscr{P}^+_l(\tau, \overline{\zeta}, \left(\begin{smallmatrix} d & 0 \\ 0 & d^{-1} \end{smallmatrix} \right), \overline{s}) \right\rangle_2 \operatorname{Im}(\tau)^l d\mu(\tau).$$

It follows from Remark 5.4, (4.45) and (6.3) that the integral in (6.4) is well-defined.

The pullback formula (5.3) combined with Theorem 6.2 and Theorem 4.10 gives rise to the before mentioned integral formula of the standard zeta function of a common Hecke eigenform f.

Theorem 6.4 *Let $l \in 2\mathbb{Z}$, $l \geq 3$, satisfy $2l + \mathrm{sig}(L) \equiv 0 \bmod 4$. Let $f \in S_l(\rho_{L,1})$ and $E_{l,0}^2$ the Eisenstein series in Definition 4.3. If $l + 2\,\mathrm{Re}(s) > 3$, then, for any $\zeta \in \mathbb{H}$,*

$$
\sum_{\lambda \in L'/L} \left(\int_{\Gamma_1 \backslash \mathbb{H}} \left\langle f(\tau) \otimes \mathfrak{e}_\lambda, E_{l,0}^2\left(\left(\begin{smallmatrix} \tau & 0 \\ 0 & -\bar{\zeta} \end{smallmatrix} \right), \bar{s} \right) \right\rangle_2 \,\mathrm{Im}(\tau)^l d\mu(\tau) \right) \mathfrak{e}_\lambda
$$
$$
= K(l,s) \sum_{d \in \mathbb{N}} d^{-l-2s} \left(f \mid_{k,L} T\left(\begin{smallmatrix} d^2 & 0 \\ 0 & 1 \end{smallmatrix} \right) \right)(\zeta),
\tag{6.6}
$$

where $K(l,s) = \dfrac{e(\mathrm{sig}(L)/8)}{|L'/L|^{1/2}}(-1)^{-s} C(l,s)$ and $C(l,s)$ is specified in (4.37). Moreover, if f is a common eigenform of all Hecke operators $T\left(\begin{smallmatrix} d^2 & 0 \\ 0 & 1 \end{smallmatrix} \right)$, the right-hand side of the above identity coincides with

$$
K(l,s) \left(\sum_{d \in \mathbb{N}} \lambda_d(f) d^{-l-2s} \right) f(\zeta).
\tag{6.7}
$$

Proof This is routine work. We have just to collect the results we established before and put them together. Equation (6.5) in Remark 6.3 combined with Theorems 6.2 and 4.10 allows us to replace the left-hand side of the above stated identity by

$$
\frac{e(\mathrm{sig}(L)/8)}{|L'/L|^{1/2}}(-1)^{-s} C(l,s) \sum_{d \in \mathbb{N}} \frac{g_d(L)}{g(L)} d^{-l-2s} \left(f \mid_{k,L} T\left(\begin{smallmatrix} d & 0 \\ 0 & d^{-1} \end{smallmatrix} \right) \right)(\zeta).
$$

Employing the relation (4.8) afterwards, gives the desired result.   □

We are now in a position to prove a result concerning the analytic properties of $Z(s,f)$. Corollary 4.6 together with (6.6) gives us the means to transfer the analytic properties of $E_{l,0}^2$ to the standard zeta function. We use the same notation as in Corollary 4.6.

Theorem 6.5 *Let $l \in 2\mathbb{Z}$, $l \geq 3$, satisfy $2l + \mathrm{sig}(L) \equiv 0 \bmod 4$ and $f \in S_l(\rho_{L,1})$ a common eigenform of the Hecke operators $T\left(\begin{smallmatrix} d^2 & 0 \\ 0 & 1 \end{smallmatrix} \right)$. Then the Dirichlet series $Z(s,f)$ can be continued meromorphically to the entire complex s-plane. If additionally L'/L is anisotropic and $|L'/L|$ odd, then*

$$
\mathscr{Z}(s,f) = K(l,s) Z(2s+l,s)
$$

satisfies the following functional equation

$$
\mathscr{Z}\left(s - \frac{l}{2}, f \right) = \xi(2s - \frac{3}{2}, 2.l, P) \prod_{p \| |L'/L|} \left(C_p(\mathrm{id}, I_0) + C_p(\mathrm{id}, I_0 \cup I_1) + C_p(\sigma, I_0) \right)
$$
$$
\times \mathscr{Z}\left(\frac{3}{2} - s - \frac{l}{2}, f \right).
\tag{6.8}
$$

Proof First, we note that

$$
\overline{E_{l,0}^2\left(\left(\begin{smallmatrix} \tau & 0 \\ 0 & -\bar{\zeta} \end{smallmatrix} \right), \bar{s} \right)} = E_{l,0}^{2*}\left(\left(\begin{smallmatrix} \bar{\tau} & 0 \\ 0 & -\zeta \end{smallmatrix} \right), s \right).
$$
$$
= E_{l,0}^2\left(\left(\begin{smallmatrix} -\bar{\tau} & 0 \\ 0 & \zeta \end{smallmatrix} \right), s \right),
$$

where the last equation is due to (4.19). Clearly, this identity holds for each component function of $E_{l,0}^2$:

$$
\overline{E_{l,0}^2\left(\left(\begin{smallmatrix} \tau & 0 \\ 0 & -\bar{\zeta} \end{smallmatrix} \right), \bar{s} \right)}_\mu = E_{l,0}^2\left(\left(\begin{smallmatrix} -\bar{\tau} & 0 \\ 0 & \zeta \end{smallmatrix} \right), s \right)_\mu
\tag{6.9}
$$

for any $\mu \in (L'/L)^2$. From this follows the first assertion.

For the functional equation we make use of the fact that (4.30) is valid for each component function of $E_{l,0}^2$, which we can immediately read off the proof of Theorem 3.16 of [27] and [10], Sect. 2.2 on pp. 641 and 642. Taking this into account, we obtain the claimed functional equation.   □

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